# Auctions for Heterogeneous Items and Budget Limits

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We study individual rational, Pareto optimal, and incentive compatible mechanisms for auctions with heterogeneous items and budget limits. We consider settings with multi-unit demand and additive valuations. For single-dimensional valuations we prove a positive result for randomized mechanisms, and a negative result for deterministic mechanisms. While the positive result allows for private budgets, the negative result is for public budgets. For multi-dimensional valuations and public budgets we prove an impossibility result that applies to deterministic and randomized mechanisms. Taken together this shows the power of randomization in certain settings with heterogeneous items, but it also shows its limitations.

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## 1. INTRODUCTION

A canonical problem in mechanism design is the design of economically efficient auctions that satisfy individual rationality and incentive compatibility. When utilities are quasi-linear these goals are achieved by the Vickrey-Clarke-Groves (VCG) mechanism. In many practical situations, including settings in which the agents have budget limits, quasi-linearity is violated and, thus, the VCG mechanism is not applicable.

Ausubel [2004] describes an ascending-bid auction for homogeneous items that yields the same outcome as the sealed-bid Vickrey auction, but offers advantages in terms of simplicity, transparency, and privacy preservation. In his concluding remarks he points out that "when budgets impair the bidding of true valuations in a sealed-bid Vickrey auction, a dynamic auction may facilitate the expression of true valuations

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while staying within budget limits" (p. 1469). Dobzinski et al. [2012] show that an adaptive version of Ausubel's "clinching auction" is indeed the unique mechanism that satisfies individual rationality, Pareto optimality, and incentive compatibility in settings with *public* budgets. They use this fact to show that there can be no mechanism that achieves those properties for *private* budgets. An important restriction of Dobzinski et al.'s impossibility result for private budgets is that it only applies to *deterministic* mechanisms. In fact, as Bhattacharya et al. [2010] show, there exists a *randomized* mechanism for homogeneous items that is individual rational, Pareto optimal, and incentive compatible with private budgets.

As Ausubel [2006] points out, "situations abound in diverse industries in which heterogeneous (but related) commodities are auctioned" (p. 602). He also describes an ascending-bid auction, the "crediting and debiting auction", that takes the place of the "clinching auction" when items are heterogeneous. Positive and negative results for *deterministic* mechanisms and public budgets that apply to heterogeneous items are given in [Fiat et al. 2011; Lavi and May 2012; Goel et al. 2012; Colini-Baldeschi et al. 2012]. We focus on *randomized* mechanisms for heterogeneous items, and prove positive results for private budgets and negative results for public budgets. We thus explore the power and limitations of randomization in this setting.

### 1.1. Our Contribution

We analyze two settings with heterogeneous items and additive valuations. In the first setting the valuations are single-dimensional in that each agent has a valuation, each item has a quality, and an agent's valuation for an item is the product of the item's quality and the agent's valuation.[1] In the second setting the valuations are multi-dimensional in that each agent has an arbitrary, non-negative valuation for each item. In both cases we analyze whether a deterministic or randomized mechanism exists that satisfies individual rationality (IR), Pareto optimality (PO), and incentive compatibility (IC). For both types of mechanisms we distinguish between settings with public budgets and settings with private budgets. For randomized mechanisms the corresponding properties can either be satisfied ex interim or they can be satisfied ex post. The former requires that the property is satisfied in expectation over the outcomes the randomized mechanism produces, while the latter requires that it is satisfied by every possible outcome of the mechanism.

(a) For **single-dimensional** valuations we present a *deterministic* mechanism for *divisible* items that is IR, PO, and IC with *public* budgets and a *randomized* mechanism for both *divisible* and *indivisible* items that is IR ex interim, PO ex post, and IC ex interim with *private* budgets. These mechanisms also satisfy another desirable property, namely "no positive transfers" (NPT), which requires that the individual payments are non-negative. We obtain these mechanisms through a general reduction from the setting with multiple, heterogeneous items to the setting of a single and by definition homogeneous item. This allows us to apply the mechanisms for this setting presented in [Bhattacharya et al. 2010]. The main difficulty in showing that the resulting deterministic and randomized mechanisms for multiple items have the desired properties is to show that they satisfy PO resp. PO ex post. For this we argue that the reduction preserves a certain structural property of the mechanisms for a single item.

[1]Such valuations arise whenever the agents agree about the relative values of the items. One concrete example is an auction in which display ads are sold in bulks consisting of a certain number of impressions together with per-impression valuations. Another example are auctions in which display ads of different size are sold and the valuations are proportional to size. In both cases the respective per-item valuations are the product of the item's quality, either the number of impressions or the size, and the agent's valuation, either per impression or per pixel.

We connect this structural property to a novel "no trade" (NT) condition, and show that it is equivalent to PO resp. PO ex post.

(b) For **single-dimensional** valuations the impossibility result of Dobzinski et al. [2012] implies that there can be no deterministic mechanism for indivisible items that is IR, PO, and IC for *private* budgets. We show that for heterogeneous items there can also be no deterministic mechanism for indivisible items that is IR, PO, and IC for *public* budgets. To this end we extend the "classic" result that IC mechanisms must satisfy "value monotonicity" (VM) and "payment identity" (PI) from settings without budgets to settings with public budgets. To establish the impossibility result, we then use NT and PI to derive a lower bound on the payments that conflicts with the upper bounds on the payments required by IR. Our impossibility result is tight in the sense that if any of the conditions is relaxed such a mechanism exists: (i) For *homogeneous*, indivisible items a deterministic mechanism is given by Dobzinski et al. [2012]. (ii) For heterogeneous items we give a deterministic mechanism for *divisible* and a *randomized* mechanism for indivisible items as described above. We thus obtain a strong separation between deterministic mechanisms, that do *not* exist for *public* budgets, and randomized mechanisms, that exist for *private* budgets. This separation is stronger than in the homogeneous items setting, where a deterministic mechanism exists for public budgets.

(c) For **multi-dimensional** valuations the impossibility result of Fiat et al. [2011] implies that there can be no deterministic mechanism for *indivisible* items that is IR, PO, and IC for public budgets. We show that there can also be no deterministic mechanism with these properties for *divisible* items. To prove this we observe that—just as in settings without budgets—every mechanism that satisfies IC with public budgets must satisfy "weak monotonicity" (WMON). Then we show that in certain settings this condition will be violated. For this we use that multi-dimensional valuations enable the agents to manipulate in a sophisticated manner. While all previous impossibility results in this area used agents that either only overstate or only understate their valuations, we use an agent that overstates his valuation for some item and understates his valuation for another item. We use our impossibility result for deterministic mechanisms to show that for both divisible and indivisible items there can be no randomized mechanism that is IR ex interim, PO ex interim, and IC ex interim with public budgets. This is the first impossibility result for randomized mechanisms in this domain. It also establishes an interesting separation between multi-dimensional valuations, where no such mechanism exists, and single-dimensional valuations, where such a mechanism exists.

### 1.2. Related Work

Homogeneous items were studied by Dobzinski et al. [2012], Bhattacharya et al. [2010], and Lavi and May [2012]. Dobzinski et al. show that for both divisible and indivisible items there is a deterministic mechanism that is IR, PO, and IC with public budgets, and that no *deterministic* mechanisms can achieve this with *private* budgets. Bhattacharya et al. show that there is a randomized mechanism for both divisible and indivisible items that is IR ex interim, PO ex post, NPT ex post, and IC ex interim with private budgets. Lavi and May prove an impossibility result for non-additive valuations with decreasing marginals. The impossibility result of Dobzinski et al. applies to both of our settings, but our impossibility results are stronger as they are for *public* budgets and, in the case of multi-dimensional valuations, also apply to *randomized* mechanisms. The positive results of Dobzinski et al. and Bhattacharya et al. do *not* apply to our settings as we study *heterogeneous* items, not homogeneous items. The impossibility result of Lavi and May does *not* apply to our settings as the valuations that we study are *additive*.

Heterogeneous items were first studied by Fiat et al. [2011]. In their model each agent has the same valuation for each item in an agent-dependent interest set and zero for all other items. They give a deterministic mechanism for indivisible items that satisfies IR, NPT, PO, and IC when both interest sets and budgets are public. They also show that when the interest sets are private, then there can be no deterministic mechanism that satisfies IR, PO, and IC. The positive result of Fiat et al. does *not* apply to our settings as it is not always possible to express the valuations that we consider in terms of per-agent valuations and interest sets. The impossibility result of Fiat et al. applies to our multi-dimensional setting and shows that there can be no *deterministic* mechanism that satisfies IR, PO, and IC with public budgets for *indivisible* items. Our impossibility result for this setting is stronger as it also applies to *randomized* mechanisms and *divisible* items.

Settings with heterogeneous items were subsequently, and in parallel to this paper, studied by Colini-Baldeschi et al. [2012] and Goel et al. [2012]. The former study problems in which the agents are interested in a certain number of slots for each of a set of keywords. The slots are associated with click-through rates that are assumed to be identical across keywords. The latter study settings in which the agents have identical valuations per item but the allocations must satisfy polyhedral or polymatroidal constraints. The settings studied in these papers are more general than the single-dimensional valuations setting studied here. On the one hand this implies that our impossibility result for this setting applies to their settings, showing that in their settings there can be no deterministic mechanism for *indivisible* items that is IC with public budgets. On the other hand this implies that their positive results apply to our setting. This shows the existence of deterministic mechanisms for divisible items and randomized mechanisms for both divisible and indivisible items that are IC with *public* budgets in our setting. Our positive result for this setting is stronger as it shows the existence of a mechanism that is IC with *private* budgets. Finally, the impossibility results of Colini-Baldeschi et al. and Goel et al. either assume non-additive valuations or that the allocations satisfy arbitrary polyhedral constraints and therefore do *not* apply to the multi-dimensional valuations setting that we study here.

We summarize the results from this paper and the related work along with open problems in Figure 1.

## 2. PROBLEM STATEMENT

We are given a set $N$ of $n$ agents and a set $M$ of $m$ items. We distinguish between settings with divisible items and settings with indivisible items. In both settings we use $X = \prod_{i=1}^{n} X_i$ for the allocation space. For divisible items $X_i = [0,1]^m$ for all agents $i \in N$ and $x_{i,j} \in [0,1]$ denotes the fraction of item $j \in M$ that is allocated to agent $i \in N$. For indivisible items $X_i = \{0,1\}^m$ for all agents $i \in N$ and $x_{i,j} \in \{0,1\}$ indicates whether item $j \in M$ is allocated to agent $i \in N$ or not. In both cases we require that $\sum_{i=1}^{n} x_{i,j} \leq 1$ for all items $j \in M$. We do *not* require that $\sum_{j=1}^{m} x_{i,j} \leq 1$ for all agents $i \in N$, i.e., we do *not* assume that the agents have unit demand.

Each agent $i$ has a type $\theta_i = (v_i, b_i)$ consisting of a valuation function $v_i : X_i \to \mathbb{R}_{\geq 0}$ and a budget $b_i \in \mathbb{R}_{\geq 0}$. We use $\Theta = \prod_{i=1}^{n} \Theta_i$ for the type space. We consider two settings with heterogeneous items, one with multi- and one with single-dimensional valuations. In the first setting, each agent $i \in N$ has a valuation $v_{i,j} \in \mathbb{R}_{\geq 0}$ for each item $j \in M$ and agent $i$'s valuation for allocation $x_i$ is $v_i(x_i) = \sum_{j=1}^{m} x_{i,j} v_{i,j}$. In the second setting, each agent $i \in N$ has a valuation $v_i \in \mathbb{R}_{\geq 0}$, each item $j \in M$ has a quality $\alpha_j \in \mathbb{R}_{\geq 0}$, and agent $i$'s valuation for allocation $x_i \in X_i$ is $v_i(x_i) = \sum_{j=1}^{m} x_{i,j} \alpha_j v_i$. For simplicity we will assume that in this setting $\alpha_1 > \alpha_2 > \cdots > \alpha_m$ and that $v_1 > v_2 > \cdots > v_n > 0$.

| | | homogeneous | | heterogeneous & additive | | | |
|---|---|---|---|---|---|---|---|
| | budgets | add. | non-add. | interest set public/private | multi-keyw. unit demand | single-dim. | multi-dim. |
| det. | public | + [D] | − [L,C] | + [F] / − [F] | ⊖ | ⊖ | − [F] |
| | private | − [D] | − [D] | − [D] / − [D] | − [D] | − [D] | − [D] |
| rand. | public | + [D] | ? | + [F] / ? | + [C,G] | ⊕ | ⊖ |
| | private | + [Bh] | ? | ? / ? | ? | ⊕ | ⊖ |

| | | homogeneous | | heterogeneous & additive | | | |
|---|---|---|---|---|---|---|---|
| | budgets | add. | non-add. | polymatroid constraints | multi-keyw. unit demand | single-dim. | multi-dim. |
| det. | public | + [D,Bh] | −[G] | + [G] | + [C,G] | ⊕ | ⊖ |
| | private | − [D] | − [D] | − [D] | − [D] | − [D] | − [D] |
| rand. | public | + [D,Bh] | ? | +[G] | + [C,G] | ⊕ | ⊖ |
| | private | + [Bh] | ? | ? | ? | ⊕ | ⊖ |

Fig. 1. Summary of the results for indivisible items (upper table) and divisible items (lower table). A plus (+ or ⊕) indicates a positive result, and a minus (− or ⊖) indicates a negative result. We use + and − for results from the related work with abbreviated references in brackets, and ⊕ and ⊖ for results from this paper. A question mark (?) indicates that nothing is known for this setting. For the model with interest sets the table has two entries, one for public and one for private interest sets.

A (direct) mechanism $\mathcal{M} = (x, p)$ consisting of an allocation rule $x : \Theta \to X$ and a payment rule $p : \Theta \to \mathbb{R}^n$ is used to compute an outcome $(x, p)$ consisting of an allocation $x \in X$ and payments $p \in \mathbb{R}^n$. A mechanism is deterministic if the computation of $(x, p)$ is deterministic, and it is randomized if the computation of $(x, p)$ is randomized. We allow the resulting allocation and payments to be arbitrarily correlated.

We assume that the agents are utility maximizers and as such need not report their types truthfully. We consider settings in which both the valuations and budgets are private and settings in which only the valuations are private and the budgets are public. When the budgets are public then they are known to the auctioneer and all agents. Private valuations/budgets mean that only the agent itself knows its valuation/budget, but not the other agents or the auctioneer. In the private values and private budgets setting a report by agent $i \in N$ with true type $\theta_i = (v_i, b_i)$ can be any type $\theta'_i = (v'_i, b'_i)$. In the private values but public budgets setting agent $i \in N$ is restricted to reports of the form $\theta'_i = (v'_i, b_i)$. In both settings, if mechanism $\mathcal{M} = (x, p)$ is used to compute an outcome for reported types $\theta' = (\theta'_1, \ldots, \theta'_n)$ and the true types are $\theta = (\theta_1, \ldots, \theta_n)$ then the utility of agent $i \in N$ is

$$u_i(x_i(\theta'), p_i(\theta'), \theta_i) = \begin{cases} v_i(x_i(\theta')) - p_i(\theta') & \text{if } p_i(\theta') \le b_i, \text{ and} \\ -\infty & \text{otherwise.} \end{cases}$$

For deterministic mechanisms and their outcomes we are interested in the following properties:

(a) *Individual rationality (IR)*: A mechanism is IR if it always produces an IR outcome. An outcome $(x, p)$ for types $\theta = (v, b)$ is IR if it is (i) *agent rational:* $u_i(x_i, p_i, \theta_i) \ge 0$ for all agents $i \in N$ and (ii) *auctioneer rational:* $\sum_{i=1}^n p_i \ge 0$.

(b) *Pareto optimality (PO)*: A mechanism is PO if it always produces a PO outcome. An outcome $(x, p)$ for types $\theta = (v, b)$ is PO if there is no other outcome $(x', p')$ such that $u_i(x'_i, p'_i, \theta_i) \ge u_i(x_i, p_i, \theta_i)$ for all agents $i \in N$ and $\sum_{i=1}^n p'_i \ge \sum_{i=1}^n p_i$, with at least one of the inequalities strict.[2] Note that we do not explicitly require that the

[2]Both IR and PO are defined with respect to the reported types, and are satisfied with respect to the true types only if the mechanism also satisfies IC.

alternate outcome is IR, but that only IR outcomes can dominate an IR outcome. That means that if we consider a PO *and* IR outcome then the two definitions are actually equivalent.

(c) *No positive transfers (NPT)*: A mechanism satisfies NPT if it always produces an NPT outcome. An outcome $(x, p)$ satisfies NPT if $p_i \geq 0$ for all agents $i \in N$.

(d) *Incentive compatibility (IC)*: A mechanism satisfies IC if for all agents $i \in N$, all true types $\theta$, and all reported types $\theta'$ we have $u_i(x_i(\theta_i, \theta'_{-i}), p_i(\theta_i, \theta'_{-i}), \theta_i) \geq u_i(x_i(\theta'_i, \theta'_{-i}), p_i(\theta'_i, \theta'_{-i}), \theta_i)$.

For randomized mechanisms we are naturally interested in randomized outcomes, which are distributions over deterministic ones. We then consider the expected utility an agent gets and compare it to the expected utility that the agent could get with other randomized outcomes. If a randomized outcome satisfies the above conditions in this way, we say it satisfies them *ex interim*. Alternatively, if each deterministic outcome in the support of a randomized outcome has this property, we say it satisfies the property *ex post*. For outcomes that are IR ex interim and PO ex interim only outcomes that are IR ex interim can be better. Hence our negative results for randomized mechanisms also apply under this alternate definition.

## 3. SINGLE-DIMENSIONAL VALUATIONS

In this section we present exact characterizations of PO resp. PO ex post outcomes and deterministic mechanisms that are IC with public budgets. We characterize PO resp. PO ex post by a simpler "no trade" condition and extend the "classic" characterization results for deterministic mechanisms for single-dimensional valuations (see, e.g., [Myerson 1981; Archer and Tardos 2001]) that are IC without budgets to settings with public budgets. We then show our main positive result, i.e., the existence of *randomized* mechanisms for divisible and indivisible items that are IR ex interim, PO *ex post*, and IC ex interim for *private* budgets. We complement this positive result with an impossibility result for *deterministic* mechanisms for indivisible items that applies even when budgets are *public*.

### 3.1. Exact Characterizations of Pareto Optimality and Incentive Compatibility

We start by characterizing PO resp. PO ex post outcomes through a simpler "no trade" condition. In the deterministic setting we consider an outcome $(x, p)$ and compare it to alternate allocations $x'$. In the randomized setting we consider a deterministic outcome and compare it to possibly randomized allocations $x'$. In what follows we use $x'_{i,j}$ to denote the expected fraction of item $j$ agent $i$ gets. This allows us to treat the two settings in a unified manner.

We say that an outcome $(x, p)$ for single-dimensional valuations satisfies *no trade (NT)* if (a) $\sum_{i \in N} x_{i,j} = 1$ for all $j \in M$, and (b) there is no allocation $x'$ such that for $\delta_i = \sum_{j \in M}(x'_{i,j} - x_{i,j})\alpha_j$ for all $i \in N$, $W = \{i \in N \mid \delta_i > 0\}$, and $L = \{i \in N \mid \delta_i \leq 0\}$ we have $\sum_{i \in N} \delta_i v_i > 0$ and $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \geq 0$. The quantity $\delta_i v_i$ is how much valuation agent $i$ gains/loses when switching from allocation $x$ to $x'$. The agents in $W$ are "winners", while the agents in $L$ are "losers". Winners are willing to increase their payment by at most $\min(b_i - p_i, \delta_i v_i)$, while losers would need to be paid $\delta_i v_i$. The definition says that there should be no alternative assignment that strictly increases the sum of the valuations and allows the winners to compensate the losers.

Here is an example: Consider a setting with two agents and a single indivisible item. Suppose that the agents have valuations $10$ and $5$ and budgets $6$ and $4$. Then the outcome $(x, p)$ which gives the item to agent $2$ at a price of $4$ does *not* satisfy NT. This is because the alternate allocation $x'$ which gives the item to agent $1$ has $\delta_1 v_1 = 1 \cdot 10 = 10$ and $\delta_2 v_2 = -1 \cdot 5 = -5$ and thus $\sum_{i \in N} \delta_i v_i > 0$. Moreover $W = \{1\}$ and $L = \{2\}$ and

$\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i = \min(6, 10) - 5 \geq 0$. Indeed we could re-assign the item from agent 2 to agent 1, increase agent 1's payment by 5, and decrease agent 2's payment by 5. In the resulting outcome agent 1 would have a strictly higher utility, agent 2's utility would be unchanged, and the sum of payments would increase by one. Hence the original outcome was not PO.

PROPOSITION 3.1. *An outcome $(x, p)$ for single-dimensional valuations and either divisible or indivisible items that respects the budget limits is PO resp. PO ex post if and only if it satisfies NT.*

Next we characterize deterministic mechanisms for indivisible items that are IC with public budgets by "value monotonicity" and "payment identity". A deterministic mechanism $\mathcal{M} = (x, p)$ for single-dimensional valuations and indivisible items that respects the publicly known budgets satisfies *value monotonicity (VM)* if for all $i \in N$, $\theta_i = (v_i, b_i)$, $\theta'_i = (v'_i, b_i)$, and $\theta_{-i} = (v_{-i}, b_{-i})$ we have that $v_i \leq v'_i$ implies $\sum_{j \in M} x_{i,j}(\theta_i, \theta_{-i})\alpha_j \leq \sum_{j \in M} x_{i,j}(\theta'_i, \theta_{-i})\alpha_j$. A deterministic mechanism $\mathcal{M} = (x, p)$ for single-dimensional valuations and indivisible items that respects the publicly known budgets satisfies *payment identity (PI)* if for all $i \in N$ and $\theta = (v, b)$ with $c_{\gamma_t} \leq v_i < c_{\gamma_{t+1}}$ we have $p_i(\theta) = p_i((0, b_i), \theta_{-i}) + \sum_{s=1}^{t}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(b_i, \theta_{-i})$, where $\gamma_0 < \gamma_1 < \ldots$ are the values $\sum_{j \in M} x_{i,j}\alpha_j$ can take and $c_{\gamma_s}(b_i, \theta_{-i})$ for $1 \leq s \leq t$ are the corresponding critical valuations. While VM ensures that stating a higher valuation can only lead to a better allocation, PI gives a formula for the payment in terms of the possible allocations and the critical valuations.

PROPOSITION 3.2. *A deterministic mechanism $\mathcal{M} = (x, p)$ for single-dimensional valuations and indivisible items that respects the publicly known budgets is IC if and only if it satisfies VM and PI.*

### 3.2. Randomized Mechanisms for Indivisible and Divisible Items

We obtain our positive result through a reduction to the setting with a single (and thus homogeneous) item that allows us to apply the following proposition from [Bhattacharya et al. 2010]. The basic building block of the mechanisms mentioned in this proposition is the "adaptive clinching auction" for a single divisible item. It is described for two agents in [Dobzinski et al. 2012] and as a "continuous time process" for arbitrarily many agents in [Bhattacharya et al. 2010].

PROPOSITION 3.3 ([BHATTACHARYA ET AL. 2010]). *For a single divisible item there exists a deterministic mechanism that satisfies IR, NPT, PO, and IC for public budgets. Additionally, for a single divisible or indivisible item there exists a randomized mechanism that satisfies IR ex interim, NPT ex post, PO ex post, and IC ex interim for private budgets.*

For indivisible items we reduce the multi-item to the single-item setting by applying the *randomized* mechanism for a single *indivisible* item of Bhattacharya et al. [2010] to a single indivisible item for which agent $i \in N$ has valuation $\tilde{v}_i = \sum_{j \in M} \alpha_j v_i$. We then map the single-item outcome $(\tilde{x}, \tilde{p})$ into an outcome $(x, p)$ for the multi-item setting by setting $x_{i,j} = 1$ for all $j \in M$ if and only if $\tilde{x}_i = 1$ and setting $p_i = \tilde{p}_i$ for all $i \in N$.

A similar reduction works in the case of divisible items. The only difference is that in this case we use the *deterministic* or *randomized* mechanisms of Bhattacharya et al. [2010] for a single *divisible* item, and then map the single-item outcome $(\tilde{x}, \tilde{p})$ into a multi-item outcome by setting $x_{i,j} = \tilde{x}_i$ for all $i \in N$ and all $j \in M$ and setting $p_i = \tilde{p}_i$ for all $i \in N$.

The main difficulty in proving that the resulting mechanisms have the claimed properties is to establish that they are PO/PO ex post. For this we argue that these particular ways of mapping the single-item outcome into a multi-item outcome preserves a specific structural property of the single-item outcome which remains to be sufficient for PO/PO ex post also in the multi-item setting.

PROPOSITION 3.4. *For indivisible or divisible items, if $(\tilde{x}, \tilde{p})$ denotes the randomized outcome for a single item of the randomized mechanism of Bhattacharya et al. [2010] and $(\bar{x}, \bar{p})$ denotes the randomized outcome for the multi-item setting constructed as described above, then $\mathrm{E}\left[u_i(\bar{x}_i, \bar{p}_i)\right] = \mathrm{E}\left[u_i(\tilde{x}_i, \tilde{p}_i)\right]$ for all $i \in N$. Similarly, for divisible items, if $(\tilde{x}, \tilde{p})$ denotes the deterministic outcome for a single item of the deterministic mechanism of Bhattacharya et al. [2010] and $(\bar{x}, \bar{p})$ denotes the deterministic outcome for the multi-item setting constructed as described above, then $u_i(\bar{x}_i, \bar{p}_i) = u_i(\tilde{x}_i, \tilde{p}_i)$ for all $i \in N$.*

THEOREM 3.5. *For single-dimensional valuations, divisible or indivisible items, and private budgets there is a randomized mechanism that satisfies IR ex interim, NPT ex post, PO ex post, and IC ex interim. Additionally, for single-dimensional valuations and divisible items there is a deterministic mechanism that satisfies IR, NPT, PO, and IC for public budgets.*

PROOF. IR resp. IR ex interim and IC resp. IC ex interim follow from Proposition 3.4 and the fact that the corresponding mechanisms of Bhattacharya et al. [2010] are IR resp. IR ex interim and IC resp. IC ex interim. NPT resp. NPT ex post follows from the fact that the payments in our mechanisms and the mechanisms of Bhattacharya et al. [2010] are the same, and the mechanisms in [Bhattacharya et al. 2010] satisfy NPT resp. NPT ex post. For PO (ex post) we argue that the structural property of the outcomes of the mechanisms in [Bhattacharya et al. 2010] that (a) $\sum_{i \in N} \tilde{x}_{i,j} = 1$ for all $j \in M$ and (b) $\sum_{j \in M} \tilde{x}_{i,j} > 0$ and $\tilde{v}_{i'} > \tilde{v}_i$ imply $\tilde{p}_{i'} = b_{i'}$ (both ex post) is preserved by the mapping to the multi-item setting and remains to be sufficient for PO (ex post).

We first show that the property is preserved. For this observe that $\sum_{i \in N} \tilde{x}_{i,j} = 1$ for all $j \in M$ implies that $\sum_{i \in N} x_{i,j} = 1$ for all $j \in M$ and that $\sum_{j \in M} \tilde{x}_{i,j} > 0$ and $\tilde{v}_{i'} > \tilde{v}_i$ imply $\tilde{p}_{i'} = b_{i'}$ implies that $\sum_{j \in M} x_{i,j} > 0$ and $v_{i'} > v_i$ imply $p_{i'} = b_{i'}$.

Next we show that the property remains to be sufficient for PO (ex post). For this assume by contradiction that the outcome $(x, p)$ is *not* PO (ex post). Then, by Proposition 3.1, there exists a (possibly randomized) $x'$ such that $\sum_{i \in N} \delta_i v_i > 0$ and $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \geq 0$, where $\delta_i = \sum_{j \in M}(x'_{i,j} - x_{i,j})\alpha_j$, $W = \{i \in N \mid \delta_i > 0\}$, and $L = \{i \in N \mid \delta_i \leq 0\}$.

Because $(x, p)$ satisfies condition (a), i.e., $\sum_{i \in N} x_{i,j} = 1$ for all $j \in M$, and $x'$ is a valid assignment, i.e., $\sum_{i \in N} x'_{i,j} \leq 1$ for all $j \in M$, we have $\sum_{i \in N} \delta_i = \sum_{j \in M} \sum_{i \in N}(x'_{i,j} - x_{i,j})\alpha_j \leq 0$. Because $\sum_{i \in N} \delta_i v_i > 0$ we have $\sum_{i \in W} \delta_i v_i \geq \sum_{i \in N} \delta_i v_i > 0$ and, so, $\sum_{i \in W} \delta_i > 0$. We conclude that $\sum_{i \in L} \delta_i = \sum_{i \in N} \delta_i - \sum_{i \in W} \delta_i < 0$ and, so, $\sum_{i \in L} \delta_i v_i < 0$.

Because $(x, p)$ satisfies condition (b), i.e., $\sum_{j \in M} x_{i,j} > 0$ and $v_{i'} > v_i$ imply $p_{i'} = b_{i'}$, there exists a $t$ with $1 \leq t \leq n$ such that (1) $\sum_{j \in M} x_{i,j} \geq 0$ and $p_i = b_i$ for $1 \leq i \leq t$, (2) $\sum_{j \in M} x_{i,j} \geq 0$ and $p_i \leq b_i$ for $i = t+1$, and (3) $\sum_{j \in M} x_{i,j} = 0$ and $p_i \leq b_i$ for $t+2 \leq i \leq n$.

We complete the proof by distinguishing three cases, and showing that in each of the three cases we get a contradiction.

*Case 1:* $t = n$. Then $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) = 0$ and, thus, $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i < 0$.

*Case 2:* $t < n$ and $W \cap \{1, \dots, t\} = \emptyset$. Then $\sum_{i \in W} \delta_i v_i \leq \sum_{i \in W} \delta_i v_{t+1}$ and $\sum_{i \in L} \delta_i v_i \leq \sum_{i \in L} \delta_i v_{t+1}$ and, thus, $\sum_{i \in N} \delta_i v_i = \sum_{i \in W} \delta_i v_i + \sum_{i \in L} \delta_i v_i \leq \sum_{i \in N} \delta_i v_{t+1} \leq 0$.

*Case 3:* $t < n$ and $W \cap \{1, \ldots, t\} \neq \emptyset$. Then $\sum_{i \in W} \min(p_i - b_i, \delta_i v_i) \leq \sum_{i \in W \setminus \{1..t\}} \delta_i v_{t+1}$ and $\sum_{i \in L} \delta_i v_i \leq \sum_{i \in L} \delta_i v_{t+1}$ and, thus, $\sum_{i \in W} \min(p_i - b_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \leq (\sum_{i \in N} \delta_i - \sum_{i \in W \cap \{1,\ldots,t\}} \delta_i) v_{t+1} < 0$. □

### 3.3. Deterministic Mechanisms for Indivisible Items.

The proof of our impossibility result uses the characterizations of PO outcomes and mechanisms that are IC with public budgets as follows: (a) PO is characterized by NT and NT induces a lower bound on the agents' payments for a *specific* assignment, namely for the case that agent 1 only gets item $m$. (b) IC, in turn, is characterized by VM and PI. Now VM and PI can be used to extend the lower bound on the payments for the *specific* assignment to *all* possible assignments. (c) Finally, IR implies upper bounds on the payments that, with a suitable choice of valuations, conflict with the lower bounds on the payments induced by NT, VM, and PI.

THEOREM 3.6. *For single-dimensional valuations, indivisible items, and public budgets there can be no deterministic mechanism $\mathcal{M} = (x, p)$ that satisfies IR, PO, and IC.*

PROOF. For a contradiction suppose that there is a mechanism $\mathcal{M} = (x, p)$ that is IR, PO, and IC for all $n$ and all $m$. Consider a setting with $n = 2$ agents and $m = 2$ items in which $v_1 > v_2 > 0$ and $b_1 > \alpha_1 v_2$.

Observe that if agent 1's valuation was $v_1' = 0$ and he reported his valuation truthfully, then since $\mathcal{M}$ satisfies IR his utility would be $u_1((0, b_1), \theta_{-1}, (0, b_1)) = -p_1((0, b_1), \theta_{-1}) \geq 0$. This shows that $p_1((0, b_1), \theta_{-1}) \leq 0$.

By PO, which by Proposition 3.1 is characterized by NT, agent 1 with valuation $v_1 > v_2$ and budget $b_1 > \alpha_1 v_2$ must win at least one item because otherwise he could buy any item from agent 2 and compensate him for his loss.

PO, respectively NT, also implies that agent 1's payment for item 2 must be strictly larger than $b_1 - (\alpha_1 - \alpha_2) v_2$ because otherwise he could trade item 2 against item 1 and compensate agent 2 for his loss.

By IC, which by Proposition 3.2 is characterized by VM and PI, agent 1's payment for item 2 is given by $p_1(\{2\}) = p_1((0, b_1), \theta_{-1}) + \alpha_2 c_{\alpha_2}(b_1, \theta_{-1})$, where $c_{\alpha_2}$ is the critical valuation for winning item 2. Together with $p_1(\{2\}) > b_1 - (\alpha_1 - \alpha_2) v_2$ this shows that $c_{\alpha_2}(b_1, \theta_{-1}) > (1/\alpha_2)[b_1 - (\alpha_1 - \alpha_2) v_2 - p_1((0, b_1), \theta_{-1})]$.

IC, respectively VM and PI, also imply that agent 1's payment for any non-empty set of items $S$ in terms of the fractions $\gamma_t = \sum_{j \in S} \alpha_j > \cdots > \gamma_1 = \alpha_2 > \gamma_0 = 0$ and corresponding critical valuations $c_{\gamma_t}(b_1, \theta_{-1}) \geq \cdots \geq c_{\gamma_1}(b_1, \theta_{-1}) = c_{\alpha_2}(b_1, \theta_{-1})$ is $p_1(S) = p_1((0, b_1), \theta_{-1}) + \sum_{s=1}^{t} (\gamma_s - \gamma_{s-1}) c_{\gamma_s}(b_1, \theta_{-1})$. Because $c_{\gamma_s}(b_1, \theta_{-1}) \geq c_{\alpha_2}(b_1, \theta_{-1})$ for all $s$ and $\sum_{s=1}^{t} (\gamma_s - \gamma_{s-1}) = \sum_{j \in S} \alpha_j$ we obtain $p_1(S) \geq p_1((0, b_1), \theta_{-1}) + (\sum_{j \in S} \alpha_j) c_{\alpha_2}(b_1, \theta_{-1})$.

Combining this lower bound on $p_1(S)$ with the lower bound on $c_{\alpha_2}(b_1, \theta_{-1})$ shows that $p_1(S) > (\sum_{j \in S} \alpha_j / \alpha_2)[b_1 - (\alpha_1 - \alpha_2) v_2]$.

For $v_1$ such that $(1/\alpha_2)[b_1 - (\alpha_1 - \alpha_2) v_2] > v_1 > v_2$ we know that agent 1 must win some item, but for any non-empty set of items $S$ the lower bound on agent 1's payment for $S$ contradicts IR. □

## 4. MULTI-DIMENSIONAL VALUATIONS

In this section we obtain a partial characterization of deterministic mechanisms that are IC with public budgets by generalizing the "weak monotonicity" condition of Bikhchandani et al. [2006] from settings without budgets to settings with budgets. We use this partial characterization together with a sophisticated misreport, in which an agent understates his valuation for some item and overstates his valuation for another item, to prove that there can be no *deterministic* mechanism for *divisible* items that is IR, PO, and IC with *public* budgets. Afterwards, we use this result to show that there can be no *randomized* mechanism for either *divisible* or *indivisible* items that is IR ex interim, PO ex interim, and IC ex interim for *public* budgets.

### 4.1. Partial Characterization of Incentive Compatibility

For settings *without budgets* every deterministic mechanism that is incentive compatible must satisfy what is known as *weak monotonicity (WMON)*, namely if $x'_i$ and $x_i$ are the assignments of agent $i$ for reports $v'_i$ and $v_i$, then the difference in the valuations for the two assignments must be at least as large under $v'_i$ as under $v_i$, i.e., $v'_i(x_i(\theta'_i,\theta_{-i})) - v'_i(x_i(\theta_i,\theta_{-i})) \geq v_i(x_i(\theta'_i,\theta_{-i})) - v_i(x_i(\theta_i,\theta_{-i}))$. We show that this is also true for deterministic mechanisms that respect the public budgets.

PROPOSITION 4.1. *If a deterministic mechanism $\mathcal{M} = (x, p)$ for multi-dimensional valuations and either divisible or indivisible items that respects the publicly known budget limits is IC, then it satisfies WMON.*

### 4.2. Deterministic Mechanisms for Divisible Items

We prove the impossibility result by analyzing a setting with two agents and two items. This restriction is without loss of generality as the impossibility result for an arbitrary number of agents $n > 2$ and an arbitrary number of items $m > 2$ follows by setting $v_{i,j} = 0$ if $i > 2$ or $j > 2$. In our impossibility proof agent 2 is not budget restricted (i.e., $b_2 > v_{2,1} + v_{2,2}$). Agents can misreport their valuations, and it is not sufficient to study a single input to prove the impossibility. Hence, we study the outcome for three related cases, namely Case 1 where $v_{1,1} < v_{2,1}$ and $v_{1,2} < v_{2,2}$; Case 2 where $v_{1,1} > v_{2,1}$, $v_{1,2} < v_{2,2}$, and $b_1 > v_{1,1}$; and Case 3 where $v_{1,1} > v_{2,1}$, $v_{1,2} > v_{2,2}$, and additionally, $b_1 > v_{1,1}$, $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$, and $v_{2,1} + v_{2,2} > b_1$. We give a partial characterization of those cases, which allows us to analyze the rational behavior of the agents.

Case 1 is easy: Agent 2 is not budget restricted and has the highest valuations for both items; so he will get both items. Thus in this case the utility for agent 1 is zero.

LEMMA 4.2 (CASE 1). *Given $b_2 > v_{2,1}+v_{2,2}$, $v_{2,1} > v_{1,1}$ and $v_{2,2} > v_{1,2}$, then $x_{1,1} = 0$, $x_{1,2} = 0$, $x_{2,1} = 1$, $x_{2,2} = 1$, and $u_1 = 0$ in every IR and PO outcome selected by an IC mechanism.*

In Case 2, agent 1 has the higher valuation for item 1, while agent 2 has the higher valuation for item 2. Thus, agent 1 gets item 1 and agent 2 gets item 2. Since the only difference to Case 1 is that in Case 2 $v_{1,1} > v_{2,1}$ while in Case 1 $v_{1,1} < v_{2,1}$, the critical value whether agent 2 gets item 1 or not is $v_{2,1}$, and thus in every IC mechanism, agent 1 has to pay $v_{2,1}$ and his utility is $v_{1,1} - v_{2,1}$.

LEMMA 4.3 (CASE 2). *Given $b_2 > v_{2,1} + v_{2,2}$, $v_{1,1} > v_{2,1}$, $v_{2,2} > v_{1,2}$, and $b_1 > v_{1,1}$, then $x_{1,1} = 1$, $x_{1,2} = 0$, $x_{2,1} = 0$, $x_{2,2} = 1$, and $u_1 = v_{1,1}-v_{2,1}$ in every IR and PO outcome selected by an IC mechanism.*

In Case 3, agent 1 has a higher valuation than agent 2 for both items, but he does not have enough budget to pay for both fully. In Lemma 4.4 we show that if agent 1 does not spend his whole budget ($p_1 < b_1$) he must fully receive both items (specifically $x_{1,2} = 1$), since if not, he would buy more of them. Additionally, even if he spent his budget fully (i.e., $p_1 = b_1$) his utility $u_i$, which equals $x_{1,1}v_{1,1} + x_{1,2}v_{1,2} - b_1$, must be non-negative. Since $b_1 > v_{1,1}$ this implies that $x_{1,1}$ must be 1, i.e., he must receive item 1 fully, and $x_{1,2}$ must be non-zero. Then, in Lemma 4.5, we show that actually $x_{1,2} < 1$, which, combined with Lemma 4.4, implies that $p_1 = b_1$. The fact that $x_{1,2} < 1$, i.e, that agent 1 does not fully get item 1 *and* 2 is not surprising since he does not have enough budget to outbid agent 2 on both items as $b_1 < v_{2,1} + v_{2,2}$. However, we are even able to determine the exact value of $x_{1,2}$, which is $(b_1 - v_{2,1})/v_{2,2}$.

LEMMA 4.4 (CASE 3, PART A). *Given $v_{1,1} > v_{2,1}$, $v_{1,2} > v_{2,2}$, $b_1 > v_{1,1}$, and $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$, if $p_1 < b_1$ then $x_{1,1} = 1$ and $x_{1,2} = 1$, else if $p_1 = b_1$ then $x_{1,1} = 1$ and $x_{1,2} > 0$, in every IR and PO outcome.*

LEMMA 4.5 (CASE 3, PART B). *Given $b_2 > v_{2,1} + v_{2,2}$, $v_{1,1} > v_{2,1}$, $v_{1,2} > v_{2,2}$, $b_1 > v_{1,1}$, $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$, and $v_{2,1} + v_{2,2} > b_1$, then $p_1 = b_1$ and $x_{1,2} = (b_1 - v_{2,1})/v_{2,2} < 1$ in every IR and PO outcome selected by an IC mechanism.*

We combine these characterizations of Case 3 with (a) the WMON property shown in Proposition 4.1 and (b) a sophisticated way of agent 2 to misreport: He *overstates* his value for item 1 by a value $\alpha$ and *understates* his value for item 2 by a value $0 < \beta < \alpha$, but by such small values that Case 3 continues to hold. Thus, by Lemma 4.4 $x_{2,1}$ remains 0 (whether agent 2 misreports or does not), and thus, the WMON condition implies that $x_{2,2}$ does *not* increase. However, by the dependence of $x_{1,2}$ on $v_{2,1}$ and $v_{2,2}$ shown in Lemma 4.5, $x_{1,2}$, and thus also $x_{2,2}$ changes when agent 2 misreports. This gives a contradiction to the assumption that such a mechanism exists.

THEOREM 4.6. *There is no deterministic IC mechanism for divisible items which selects for any given input with public budgets an IR and PO outcome.*

PROOF. Let us assume by contradiction that such a mechanism exists and consider an input for which $b_2 > v_{2,1} + v_{2,2}$, $v_{1,1} > v_{2,1}$, $v_{1,2} > v_{2,2}$, $b_1 > v_{1,1}$, $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$, and $v_{2,1} + v_{2,2} > b_1$ holds. Such an input exists, for example $v_{1,1} = 4$, $v_{1,2} = 5$, $v_{2,1} = 3$, and $v_{2,2} = 4$ with budgets $b_1 = 5$ and $b_2 = 8$ would be such an input. Lemma 4.4 and 4.5 imply that $x_{1,1} = 1$, $x_{2,1} = 0$, $x_{1,2} = \frac{b_1 - v_{2,1}}{v_{2,2}}$, $x_{2,2} = 1 - x_{1,2}$, and $p_1 = b_1$. Let us consider an alternative valuation by agent 2. We define $v'_{2,1} = v_{2,1} + \alpha$ and $v'_{2,2} = v_{2,2} - \beta$ for arbitrary $\alpha, \beta > 0$ and $\alpha > \beta$ which are sufficiently small such that $v_{1,1}v'_{2,2} > v_{1,2}v'_{2,1}$ holds, and we denote the fraction of item 2 assigned to agent 2 for the alternated valuations by $x'_{2,2}$. By Proposition 4.1, IC implies WMON, and therefore, $x'_{2,2}v'_{2,2} - x_{2,2}v'_{2,2} \geq x'_{2,2}v_{2,2} - x_{2,2}v_{2,2}$. It follows that $x_{2,2} \geq x'_{2,2}$, and by Lemma 4.5, $\frac{b_1 - v_{2,1}}{v_{2,2}} \leq \frac{b_1 - v'_{2,1}}{v'_{2,2}}$. Hence, the budget of agent 1 has to be large enough, such that $b_1 \geq \frac{v_{2,2}v'_{2,1} - v_{2,1}v'_{2,2}}{v_{2,2} - v'_{2,2}} = \frac{v_{2,1}\beta + v_{2,2}\alpha}{\beta} > v_{2,1} + v_{2,2}$, but $b_1 < v_{2,1} + v_{2,2}$ holds by assumption. Contradiction! □

### 4.3. Randomized Mechanisms for Divisible and Indivisible Items

We exploit the fact that randomized mechanisms for both divisible and indivisible items are essentially equivalent to deterministic mechanisms for divisible items.

We show that for agents with budget constraints every randomized mechanism $\bar{\mathcal{M}} = (\bar{x}, \bar{p})$ for divisible or indivisible items can be mapped bidirectionally to a deterministic mechanism $\mathcal{M} = (x, p)$ for divisible items with identical expected utility for all the agents and the auctioneer when the same reported types are used as input. To turn a randomized mechanism for *divisible* or *indivisible* items into a deterministic mechanism for *divisible* items simply compute the expected values of $p_i$ and $x_{i,j}$ for all $i$ and $j$ and return them. To turn a deterministic mechanism for *divisible* items into a randomized mechanism for *divisible* or *indivisible* items simply assign the items with probability $x_{i,j}$ and keep the same payment as the deterministic mechanism.

PROPOSITION 4.7. *Every randomized mechanism $\bar{\mathcal{M}} = (\bar{x}, \bar{p})$ for agents with finite budgets, a rational auctioneer, and a limited amount of divisible or indivisible items can* *be mapped bidirectionally to a deterministic mechanism* $\mathcal{M} = (x, p)$ *for divisible items such that* $u_i(x_i(\theta'), p_i(\theta'), \theta_i) = \mathrm{E}\left[u_i(\bar{x}_i(\theta'), \bar{p}_i(\theta'), \theta_i)\right]$ *and* $\sum_{i \in N} p_i(\theta') = \mathrm{E}\left[\sum_{i \in N} \bar{p}_i(\theta')\right]$ *for all agents* $i$*, all true types* $\theta = (v, b)$*, and reported types* $\theta' = (v', b')$*.*

PROOF. Let us map $\bar{\mathcal{M}} = (\bar{x}, \bar{p})$ to $\mathcal{M} = (x, p)$ that assigns for each agent $i \in N$ and item $j \in M$ a fraction of $\mathrm{E}\left[\bar{x}_{i,j}\right]$ of item $j$ to agent $i$, and makes each agent $i \in N$ pay $\mathrm{E}\left[\bar{p}_i\right]$. The expectations exist since the feasible fractions of items and the feasible payments have an upper bound and a lower bound. For the other direction, we map $\mathcal{M} = (x, p)$ to $\bar{\mathcal{M}} = (\bar{x}, \bar{p})$ that randomly picks for each item $j \in M$ an agent $i \in N$ to which it assigns item $j$ in a way such that agent $i$ is picked with probability $x_{i,j}$, and makes each agent $i \in N$ pay $p_i$. Since $x = \mathrm{E}\left[\bar{x}\right]$ and $p = \mathrm{E}\left[\bar{p}\right]$, $\sum_{j \in M}(x_{i,j} v_{i,j}) - p_i = \mathrm{E}\left[\sum_{j \in M}(\bar{x}_{i,j} v_{i,j}) - \bar{p}_i\right]$ for all $i \in N$ and $\sum_{i \in N} p_i = \mathrm{E}\left[\sum_{i \in N} \bar{p}_i\right]$. □

This proposition implies the non-existence of randomized mechanisms stated in Theorem 4.8.

THEOREM 4.8. *There can be no randomized mechanism for divisible or indivisible items that is IR ex interim, PO ex interim, and IC ex interim, and that satisfies the public budget constraint ex post.*

PROOF. For a contradiction suppose that there is such a randomized mechanism. Then, by Proposition 4.7, there must be a deterministic mechanism for divisible items and public budgets that satisfies IR, PO, and IC. This gives a contradiction to Theorem 4.6. □

## 5. CONCLUSION AND FUTURE WORK

In this paper we analyzed IR, PO, and IC mechanisms for settings with heterogeneous items. Our main accomplishments are: (a) *Randomized* mechanisms that achieve these properties for *private* budgets and a restricted class of additive valuations. (b) An impossibility result for *randomized* mechanisms and *public* budgets for additive valuations. We are able to circumvent the impossibility result in the restricted setting because our argument for the impossibility result is based on the ability of an agent to overstate his valuation for one and understate his valuation for another item, which is not possible in the restricted setting. A promising direction for future work is to identify other valuations for which this is the case.

## APPENDIXES

## A. PROOF OF PROPOSITION 3.1

We show the claim for the deterministic setting. The claim for the randomized setting follows by interpreting $x'_{i,j}$ as the the expected fraction of item $j$ allocated to agent $i$, $p'_i$ as agent $i$'s expected payment, and $u'_i$ as its expected utility.

First we show that if $(x, p)$ satisfies PO, then it satisfies NT. To this end we show that if $(x, p)$ does *not* satisfy NT, then it is *not* PO.

Case 1: ¬ NT because ¬ (a). We can assign the unassigned fraction of the item $j \in M$ for which $\sum_{i \in N} x_{i,j} < 1$ to any agent $i \in N$ to get a contradiction to PO.

Case 2: ¬ NT because ¬ (b). There exists an assignment $x'$ such that $\sum_{i \in N} \delta_i v_i > 0$ and $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \geq 0$. Consider the outcome $(x', p')$ for which $p'_i = p_i + \min(b_i - p_i, \delta_i v_i)$ for all agents $i \in W$ and $p'_i = p_i + \delta_i v_i$ for all agents $i \in L$.

For all agents $i \in N$ we have $u'_i \geq u_i$ because

$$u'_i = \sum_{j \in M} x_{i,j} \alpha_j v_i + \delta_i v_i - p_i - \min(b_i - p_i, \delta_i v_i) \geq u_i \qquad \text{for } i \in W, \text{ and} \tag{1}$$
$$u'_i = \sum_{j \in M} x_{i,j} \alpha_j v_i + \delta_i v_i - p_i - \delta_i v_i = u_i \qquad \text{for } i \in L.$$

For the auctioneer we have $\sum_{i \in N} p'_i \geq \sum_{i \in N} p_i$ because

$$\sum_{i \in N} p'_i - \sum_{i \in N} p_i = \sum_{i \in W} p'_i + \sum_{i \in L} p'_i - \sum_{i \in N} p_i = \sum_{i \in W} (p_i + \min(b_i - p_i, \delta_i v_i)) + \sum_{i \in L} (p_i + \delta_i v_i) - \sum_{i \in N} p_i = \sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \geq 0. \tag{2}$$

If $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i > 0$, then inequality (2) is strict showing that $\sum_{i \in N} p'_i > \sum_{i \in N} p_i$. Otherwise, $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i = 0$, and since $\sum_{i \in N} \delta_i v_i > 0$ we must have $b_i - p_i < \delta_i v_i$ for at least one agent $i \in W$. For this agent $i$ inequality (1) is strict showing that $u'_i > u_i$. Hence in both cases $(x, p)$ is *not* PO.

Next we show that if $(x, p)$ satisfies NT, then it is PO. To this end we show that if $(x, p)$ is *not* PO, then it does *not* satisfy NT. If $(x, p)$ is *not* PO, then there exists an outcome $(x', p')$ such that $u'_i \geq u_i$ for all agents $i \in N$ and $\sum_{i \in N} p'_i \geq \sum_{i \in N} p_i$, with at least one of the inequalities strict.

If *not* all items are assigned completely in $(x, p)$, then we have $\neg$ (a) and so $(x, p)$ does *not* satisfy NT. Otherwise, if in $(x, p)$ all items are assigned completely, then to show that $(x, p)$ does *not* satisfy NT we have to show $\neg$ (b). To this end consider the assignment $x'$ and let $\delta_i = \sum_{j \in M} (x'_{i,j} - x_{i,j}) \alpha_j$ for $i \in N$, let $W = \{i \in N \mid \delta_i > 0\}$, and let $L = \{i \in N \mid \delta_i \leq 0\}$.

We begin by showing that $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \geq 0$. For $i \in N$ we have $p'_i - p_i \leq \min(b_i - p_i, \delta_i v_i)$ because $p'_i \leq b_i$ implies $p'_i - p_i \leq b_i - p_i$, and $u'_i \geq u_i$ implies $p'_i - p_i \leq \delta_i v_i$. It follows that $\sum_{i \in W} \min(b_i - p_i, \delta_i v_i) + \sum_{i \in L} \delta_i v_i \geq \sum_{i \in W} (p'_i - p_i) + \sum_{i \in L} (p'_i - p_i) = \sum_{i \in N} p'_i - \sum_{i \in N} p_i \geq 0$.

Next we show that $\sum_{i \in N} \delta_i v_i > 0$. Since $u'_i \geq u_i$ for all $i \in N$ we have $\sum_{i \in N} u'_i \geq \sum_{i \in N} u_i$. This implies $\sum_{i \in N} ((\sum_{j \in M} x'_{i,j} \alpha_j v_i) - p'_i) \geq \sum_{i \in N} ((\sum_{j \in M} x_{i,j} \alpha_j v_i) - p_i)$, and consequently, $\sum_{i \in N} (\sum_{j \in M} (x'_{i,j} - x_{i,j}) \alpha_j v_i) \geq \sum_{i \in N} p'_i - \sum_{i \in N} p_i$. As $\sum_{i \in N} p'_i \geq \sum_{i \in N} p_i$ it follows that

$$\sum_{i \in N} \delta_i v_i \geq \sum_{i \in N} p'_i - \sum_{i \in N} p_i \geq 0. \tag{3}$$

If $u'_i > u_i$ for some $i \in N$, then $\sum_{i \in N} u'_i > \sum_{i \in N} u_i$ and, thus, the first inequality in (3) is strict. Otherwise, if $\sum_{i \in N} p'_i > \sum_{i \in N} p_i$, then the second inequality in (3) is strict. In both cases strictness of the inequality implies that $\sum_{i \in N} \delta_i v_i > 0$.

## B. PROOF OF PROPOSITION 3.2

We begin by showing that if $\mathcal{M}$ satisfies VM and PI, then it satisfies IC. For a contradiction assume that $\mathcal{M}$ satisfies VM and PI, but that it does *not* satisfy IC. Then there exists $i \in N$, $\theta_i = (v_i, b_i)$, $\theta'_i = (v'_i, b_i)$, and $\theta_{-i} = (v_{-i}, b_{-i})$ with $v_i \neq v'_i$ such that $u_i(x_i(\theta'_i, \theta_{-i}), p(\theta'_i, \theta_{-i}), \theta_i) > u_i(x_i(\theta_i, \theta_{-i}), p(\theta_i, \theta_{-i}), \theta_i)$.

Let $c_{\gamma_t}(b_i, \theta_{-i}) \leq v_i < c_{\gamma_{t+1}}(b_i, \theta_{-i})$ and let $c_{\gamma_{t'}}(b_i, \theta_{-i}) \leq v'_i < c_{\gamma_{t'+1}}(b_i, \theta_{-i})$.

If $v_i > v_i'$ then since $\mathcal{M}$ satisfies VM and PI the utilities $u_i$ and $u_i'$ that agent $i$ gets from reports $\theta_i$ and $\theta_i'$ satisfy $u_i - u_i' = (\gamma_t - \gamma_{t'})v_i - \sum_{s=t'+1}^{t}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(b_i, \theta_{-i}) \geq (\gamma_t - \gamma_{t'})v_i - \sum_{s=t'+1}^{t}(\gamma_s - \gamma_{s-1})v_i = 0$.

If $v_i < v_i'$ then since $\mathcal{M}$ satisfies VM and PI the utilities $u_i'$ and $u_i$ that agent $i$ gets from reports $\theta_i'$ and $\theta_i$ satisfy $u_i' - u_i = (\gamma_{t'} - \gamma_t)v_i - \sum_{s=t+1}^{t'}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(b_i, \theta_{-i}) \leq (\gamma_{t'} - \gamma_t)v_i - \sum_{s=t+1}^{t'}(\gamma_s - \gamma_{s-1})v_i = 0$.

We conclude that in both cases agent $i$ is weakly better off when he reports truthfully. This contradicts our assumption that $\mathcal{M}$ does *not* satisfy IC.

Next we show that if $\mathcal{M}$ satisfies IC, then it satisfies VM. By contradiction assume that $\mathcal{M}$ satisfies IC, but that it does *not* satisfy VM. Then there exists $i \in N$, $\theta_i = (v_i, b_i)$, $\theta_i' = (v_i', b_i)$, and $\theta_{-i} = (v_{-i}, b_{-i})$ with $v_i < v_i'$ such that $\sum_{j \in M} x_{i,j}(\theta_i, \theta_{-i})\alpha_j > \sum_{j \in M} x_{i,j}(\theta_i', \theta_{-i})\alpha_j$. Since $\mathcal{M}$ satisfies IC agent $i$ with type $\theta_i$ does *not* benefit from reporting $\theta_i'$, and vice versa. Thus, $\sum_{j \in M} x_{i,j}(\theta_i, \theta_{-i})\alpha_j v_i - p_i(\theta_i, \theta_{-i}) \geq \sum_{j \in M} x_{i,j}(\theta_i', \theta_{-i})\alpha_j v_i - p_i(\theta_i', \theta_{-i})$, and $\sum_{j \in M} x_{i,j}(\theta_i', \theta_{-i})\alpha_j v_i' - p_i(\theta_i', \theta_{-i}) \geq \sum_{j \in M} x_{i,j}(\theta_i, \theta_{-i})\alpha_j v_i' - p_i(\theta_i, \theta_{-i})$. By combining these inequalities we get $(\sum_{j \in M} x_{i,j}(\theta_i, \theta_{-i})\alpha_j - \sum_{j \in M} x_{i,j}(\theta_i', \theta_{-i})\alpha_j)(v_i - v_i') \geq 0$. Since $\sum_{j \in M} x_{i,j}(\theta_i, \theta_{-i})\alpha_j > \sum_{j \in M} x_{i,j}(\theta_i', \theta_{-i})\alpha_j$ this shows that $v_i \geq v_i'$ and gives a contradiction to our assumption that $v_i < v_i'$.

We conclude the proof by showing that if $\mathcal{M}$ satisfies IC, then it satisfies PI. For a contradiction assume that $\mathcal{M}$ satisfies IC, but that it does *not* satisfy PI. Then there exists $i \in N$, $\theta_i' = (v_i', b_i)$, and $\theta_{-i} = (v_{-i}, b_{-i})$ with $c_{\gamma_{t'}} \leq v_i' < c_{\gamma_{t'+1}}$ such that $p_i(\theta_i', \theta_{-i}) \neq p_i((0, b_i), \theta_{-i}) + \sum_{s=1}^{t'}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(b_i, \theta_{-i})$, where the $\gamma_s$ are the sum over the $\alpha$'s of all possible assignments in non-increasing order and the $c_{\gamma_s}(b_i, \theta_{-i})$ are the smallest valuations (or critical valuations) that make agent $i$ win $\gamma_s$.

Consider the smallest $v_i'$ such that this is the case. For this $v_i'$ we must have $v_i' = c_{\gamma_{t'}}(b_i, \theta_{-i}) > c_{\gamma_0}(b_i, \theta_{-i}) = 0$. We must have $v_i' = c_{\gamma_{t'}}(b_i, \theta_{-i})$ because by VM agent $i$'s assignment for all reports $\theta_i'' = (v_i'', b_i)$ with $v_i''$ such that $c_{\gamma_{t'}}(b_i, \theta_{-i}) \leq v_i'' < c_{\gamma_{t'+1}}(b_i, \theta_{-i})$ is the same and, thus, by IC he must face the same payment. We must have $c_{\gamma_{t'}}(b_i, \theta_{-i}) > c_{\gamma_0}(b_i, \theta_{-i}) = 0$ because for $v_i' = 0$ we have $p(\theta_i', \theta_{-i}) = p((0, b_i), \theta_{-i})$ by definition.

Case 1: $p_i(\theta_i', \theta_{-i}) > p_i((0, b_i), \theta_{-i}) + \sum_{s=1}^{t'}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(b_i, \theta_{-i})$. Consider $\theta_i = (v_i, b_i)$ with $v_i < v_i'$ such that $c_{\gamma_{t'-1}}(b_i, \theta_{-i}) \leq v_i < c_{\gamma_{t'}}(b_i, \theta_{-i})$. Since $v_i < v_i'$ we have $p_i(\theta_i, \theta_{-i}) = p_i((0, b_i), \theta_{-i}) + \sum_{s=1}^{t'-1}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(b_i, \theta_{-i})$. If agent $i$'s type is $\theta_i'$ then for the utilities $u_i'$ and $u_i$ that he gets for reports $\theta_i'$ and $\theta_i$ we have $u_i' - u_i < (\gamma_{t'} - \gamma_{t'-1})v_i' - (\gamma_{t'} - \gamma_{t'-1})c_{\gamma_{t'}}(b_i, \theta_{-i}) = 0$. This shows that agent $i$ with type $\theta_i'$ has an incentive to misreport his type as $\theta_i$ and contradicts our assumption that $\mathcal{M}$ satisfies IC.

Case 2: $p_i(\theta_i', \theta_{-i}) < p_i((0, b_i), \theta_{-i}) + \sum_{s=1}^{t'}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(b_i, \theta_{-i})$. Let $\epsilon = p_i((0, b_i), \theta_{-i}) + \sum_{s=1}^{t'}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(b_i, \theta_{-i}) - p_i(\theta_i', \theta_{-i})$ and consider $\theta_i = (v_i, b_i)$ with $v_i < v_i'$ such that $c_{\gamma_{t'-1}}(b_i, \theta_{-i}) \leq v_i < c_{\gamma_{t'}}(b_i, \theta_{-i})$. Since $v_i < v_i'$ we have $p_i(\theta_i, \theta_{-i}) = p_i((0, b_i), \theta_{-i}) + \sum_{s=1}^{t'-1}(\gamma_s - \gamma_{s-1})c_{\gamma_s}(b_i, \theta_{-i})$. If agent $i$'s type is $\theta_i$ then for the utilities $u_i'$ and $u_i$ that he gets from reports $\theta_i'$ and $\theta_i$ we have $u_i' - u_i = (\gamma_{t'} - \gamma_{t'-1})v_i - (\gamma_{t'} - \gamma_{t'-1})c_{\gamma_{t'}}(b_i, \theta_{-i}) + \epsilon$. Since this is true for all $v_i$ with $c_{\gamma_{t'-1}}(b_i, \theta_{-i}) \leq v_i < c_{\gamma_{t'}}(b_i, \theta_{-i})$ we can choose $v_i$ such that $(\gamma_{t'} - \gamma_{t'-1})(v_i - c_{\gamma_{t'}}(b_i, \theta_{-i})) > -\epsilon$. We get $u_i' - u_i > 0$. This shows that agent $i$ with type $\theta_i$ has an incentive to misreport his type as $\theta_i'$ and contradicts our assumption that $\mathcal{M}$ satisfies IC.

### C. PROOF OF PROPOSITION 3.4

First suppose that the payments are deterministic. If $p_i > b_i$ then $\tilde{p}_i > b_i$ and $u_i(x_i, p_i, (v_i, b_i)) = u_i(\tilde{x}_i, \tilde{p}_i, (\tilde{v}_i, b_i)) = -\infty$. Otherwise, $u_i(x_i, p_i, (v_i, b_i)) = \sum_{j=1}^{m}(x_{i,j}\alpha_j v_i) - p_i = \tilde{x}_i\tilde{v}_i - \tilde{p}_i = u_i(\tilde{x}_i, \tilde{p}_i, (\tilde{v}_i, b_i))$.

Next suppose that the payments are randomized. If $\Pr[p_i > b_i] > 0$ then $\Pr[\tilde{p}_i > b_i] > 0$ and $\mathrm{E}\left[u_i(x_i, p_i, (v_i, b_i))\right] = \mathrm{E}\left[u_i(\tilde{x}_i, \tilde{p}_i, (\tilde{v}_i, b_i))\right] = -\infty$. Otherwise, $\mathrm{E}\left[u_i(x_i, p_i, (v_i, b_i))\right] = \mathrm{E}\left[\sum_{j=1}^{m}(x_{i,j}\alpha_j v_i) - p_i\right] = \mathrm{E}\left[\tilde{x}_i\tilde{v}_i - \tilde{p}_i\right] = \mathrm{E}\left[u_i(\tilde{x}_i, \tilde{p}_i, (\tilde{v}_i, b_i))\right]$.

### D. PROOF OF PROPOSITION 4.1

Fix $i \in N$ and $\theta_{-i} = (v_{-i}, b_{-i})$. By IC agent $i$ does not benefit from reporting $\theta'_i = (v'_i, b_i)$ when his true type is $\theta_i = (v_i, b_i)$, nor does he benefit from reporting $\theta_i = (v_i, b_i)$ when his true type is $\theta'_i = (v'_i, b_i)$. Thus, $v_i(x(\theta_i, \theta_{-i})) - p_i(\theta_i, \theta_{-i}) \geq v_i(x(\theta'_i, \theta_{-i})) - p_i(\theta'_i, \theta_{-i})$, and $v'_i(x(\theta'_i, \theta_{-i})) - p_i(\theta'_i, \theta_{-i}) \geq v'_i(x(\theta_i, \theta_{-i})) - p_i(\theta_i, \theta_{-i})$. By combining these inequalities we get $v'_i(x_i(\theta'_i, \theta_{-i})) - v'_i(x_i(\theta_i, \theta_{-i})) \geq v_i(x_i(\theta'_i, \theta_{-i})) - v_i(x_i(\theta_i, \theta_{-i}))$.

### E. PROOF OF LEMMA 4.2

We divide the proof into the following parts: in (a) we show that $x_{1,1} = 0$, $x_{1,2} = 0$, $x_{2,1} = 1$, and $x_{2,2} = 1$, and in (b) we show that $u_1 = 0$.

To (a): Let us assume by contradiction that we have an IR and PO outcome where $x_{1,1} > 0$ or $x_{1,2} > 0$. IR requires that $p_2 \leq x_{2,1}v_{2,1} + x_{2,2}v_{2,2}$. Hence, agent 2 can buy the fractions $x_{1,1}$ of item 1 and $x_{1,2}$ of item 2 for a payment $p$ with $x_{1,1}v_{2,1} + x_{1,2}v_{2,2} > p \geq x_{1,1}v_{1,1} + x_{1,2}v_{1,2}$ from agent 1. Because of $v_{2,1} > v_{1,1}$ and $v_{2,2} > v_{1,2}$ such a payment exists and agent 2 has enough money, since $b_2 > v_{2,1} + v_{2,2}$ implies $b_2 > v_{2,1} + v_{2,2} = (x_{1,1} + x_{2,1})v_{2,1} + (x_{1,2} + x_{2,2})v_{2,2} > p_2 + p$. The utility of agent 2 would increase and the utilities of agent 1 and the auctioneer would not decrease. Contradiction to PO!

To (b): We have already shown before that agent 1 gets no fraction of the items, and therefore, IR implies that his payments cannot be positive.

Let us consider the subcase where $v_{1,1} = v_{1,2} = 0$ and agent 1 reports truthfully. The valuations of agent 2 are positive. Because of IR the payment of agent 2 cannot exceed his reported valuation, but (a) holds when his reported valuations are positive. Therefore, agent 2 would have an incentive to understate his valuation when his payment would be positive. Hence, IR of the auctioneer implies that the payment of both agents is equal to 0. This means, that the utility of agent 1 is 0 in this case.

If there was any other reported valuation of agent 1, where he gets no items, but where his payments are negative, then he would have an incentive to lie, when his valuations are equal to 0. This would contradict IC!

### F. PROOF OF LEMMA 4.3

We divide the proof into the following parts: in (a) we show that $x_{1,1} = 1$, $x_{1,2} = 0$, $x_{2,1} = 0$, and $x_{2,2} = 1$, and in (b) we show that $u_1 = v_{1,1} - v_{2,1}$.

To (a): Let us assume by contradiction that $x_{1,2} > 0$. Then, agent 2 can buy these fractions of item 2 for a payment $p$ with $x_{1,2}v_{2,2} > p \geq x_{1,2}v_{1,2}$, which exists because of $v_{2,2} > v_{1,2}$. IR and $b_2 > v_{2,1} + v_{2,2}$ ensure that agent 2 has enough budget, since $b_2 > v_{2,1} + v_{2,2} = (x_{1,1} + x_{2,1})v_{2,1} + (x_{1,2} + x_{2,2})v_{2,2} \geq p_2 + x_{1,1}v_{2,1} + x_{1,2}v_{2,2} > p_2 + p$. The utility of the agent 2 would increase, while the utilities of agent 1 and the auctioneer would not decrease. Contradiction to PO!

Otherwise, let us assume that $x_{1,1} < 1$ and $x_{1,2} = 0$. Then, agent 1 can buy the other fractions of item 1 for a payment $p$ with $x_{2,1}v_{1,1} > p \geq x_{2,1}v_{2,1}$, which exists because of $v_{1,1} > v_{2,1}$. IR and $b_1 > v_{1,1}$ ensure that agent 1 has enough budget, since $b_1 > v_{1,1} = (x_{1,1} + x_{2,1})v_{1,1} \geq p_1 + x_{2,1}v_{1,1} > p_1 + p$. The utility of agent 1 would increase, while the utilities of agent 2 and the auctioneer would not decrease. Contradiction to PO!

To (b): We show first that $p_1 \leq v_{2,1}$. Since $x_{1,1} = 1$ and $x_{1,2} = 0$, IR requires that $p_1 \leq v_{1,1}$. If $p_1 > v_{2,1}$, then agent 1 has an incentive to lie. If he states that his valuation for item 1 is $v'_{1,1}$ with $p_1 > v'_{1,1} > v_{2,1}$, then the allocation of the items does not change, but he pays less because of IR. Contradiction to IC!

Now, we show that $p_1 \geq v_{2,1}$. Let us therefore assume by contradiction that $p_1 < v_{2,1}$. If we have $v'_{1,1}$ with $p_1 < v'_{1,1} < v_{2,1}$ instead of $v_{1,1}$, and all the other valuations are left unchanged, then Lemma 4.2 implies that $u'_1 = 0$. Hence, in this case agent 1 can increase his utility when he lies and states that his valuation is $v_{1,1}$, because his utility would be $v'_{1,1} - p_1 > 0$. Contradiction to IC!

Since agent 1 gets all fractions of item 1, no fraction of item 2, and has to pay $v_{2,1}$, his utility is $v_{1,1} - v_{2,1}$.

## G. PROOF OF LEMMA 4.4

We divide the proof into the following parts: in (a) we show that $x_{1,1} = 1$ and $x_{1,2} = 1$ if $p_1 < b_1$, in (b) we show that $x_{1,2} > (1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}}$ if $p_1 = b_1$, and in (c) we show that $x_{1,1} = 1$ and $x_{1,2} > 0$ if $p_1 = b_1$.

To (a): Let us assume by contradiction that $p_1 < b_1$ and $x_{1,j} < 1$ for an item $j \in \{1, 2\}$. Agent 1 can increase his utility by buying $\min\{\frac{b_1 - p_1}{p}, x_{2,j}\}$ fractions of item $j$ for a unit price $p$ with $v_{1,j} > p \geq v_{2,j}$ from agent 2. Such a price exists, because of $v_{1,1} > v_{2,1}$ and $v_{1,2} > v_{2,2}$. Agent 1 has enough money for the trade, since $p_1 + p \min\{\frac{b_1 - p_1}{p}, x_{2,j}\} = \min\{b_1, p_1 + p x_{2,j}\} \leq b_1$. The utility of agent 1 would increase, and the utilities of agent 2 and the auctioneer would not decrease. Contradiction to PO!

To (b): IR requires $b_1 = p_1 \leq v_{1,1}x_{1,1} + v_{1,2}x_{1,2}$, and therefore, $x_{1,2} \geq \frac{b_1 - v_{1,1}x_{1,1}}{v_{1,2}}$. If $x_{1,1} = 1$, then $b_1 > v_{1,1}$ implies that $(1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}} = 0 < \frac{b_1 - v_{1,1}}{v_{1,2}} = \frac{b_1 - v_{1,1}x_{1,1}}{v_{1,2}}$. Otherwise, if $x_{1,1} = 0$, then $b_1 > v_{1,1}$ and $v_{1,1}v_{2,2} > v_{1,2}v_{2,1}$ imply that $(1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}} = \frac{v_{2,1}}{v_{2,2}} < \frac{b_1}{v_{1,2}} = \frac{b_1 - v_{1,1}x_{1,1}}{v_{1,2}}$, and hence, $(1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}} < \frac{b_1 - v_{1,1}x_{1,1}}{v_{1,2}}$ for all $x_{1,1} \in [0, 1]$. Therefore, we have that $(1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}} < x_{1,2}$ for all possible values of $x_{1,1}$.

To (c): We split the proof into two parts. We assume by contradiction that either $p_1 = b_1$, $x_{1,1} \leq 1$ and $x_{1,2} = 0$, or that $p_1 = b_1$, $x_{1,1} < 1$ and $x_{1,2} > 0$.

Let us assume that $p_1 = b_1$, $x_{1,1} \leq 1$ and $x_{1,2} = 0$. According to $b_1 > v_{1,1}$, the utility of agent 1 is negative. Contradiction to IR!

We will now investigate the other case and assume that $p_1 = b_1$, $x_{1,1} < 1$ and $x_{1,2} > 0$. Agent 2 has the same valuation for $x_{1,2} = 1 - x_{1,1}$ fractions of item 1 and $(1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}}$ fractions of item 2. The valuation of agent 1 for $(1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}}$ fractions of item 2 is identical to the valuation for $(1 - x_{1,1})\frac{v_{2,1}v_{1,2}}{v_{2,2}v_{1,1}}$ fractions of item 1. We know that $v_{2,1}v_{1,2} < v_{2,2}v_{1,1}$. That is, that the utility of agent 1 is increased and the utilities of agent 2 and the auctioneer are not decreased, when agent 1 trades $(1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}}$ fractions of item 2 against $x_{2,1} = 1 - x_{1,1}$ fractions of item 1. Fact (b) implies that agent 1 actually has the required $(1 - x_{1,1})\frac{v_{2,1}}{v_{2,2}}$ fractions of item 2. Contradiction to PO!

## H. PROOF OF LEMMA 4.5

We divide the proof into the following parts: in (a) we show that $p_1 = b_1$ and $x_{1,2} < 1$, in (b) we show that $\frac{b_1 - v_{2,1}}{v_{2,2}} \geq x_{1,2} \geq \frac{b_1 - v_{2,1}}{v_{1,2}}$, and in (c) we show that $x_{1,2} = \frac{b_1 - v_{2,1}}{v_{2,2}}$.

To (a): Lemma 4.4 implies that the utility of agent 1 is $v_{1,1}+x_{1,2}v_{1,2}-p_1$. We know that $v_{2,1}+v_{2,2} > b_1$. Hence, we can select a sufficiently small $\epsilon > 0$ such that $v_{2,1}+v_{2,2}-\epsilon > b_1$. Because of $v_{1,1} > v_{2,1}$ and $b_1 > v_{1,1}$, we know that $v_{2,2} - \epsilon > 0$. Let us consider the case where we have $v'_{1,2}$ with $v_{2,2} > v'_{1,2} > v_{2,2} - \epsilon$ instead of $v_{1,2}$ and all other valuation are left unchanged. In this case, the utility of agent 1 is $v_{1,1} - v_{2,1}$, because of Lemma 4.3 and since $v_{2,2} > v'_{1,2}$ holds. Therefore, IC implies that

$$v_{1,1} - v_{2,1} \geq v_{1,1} + x_{1,2}v'_{1,2} - p_1. \tag{4}$$

Let us assume by contradiction that $x_{1,2} = 1$, then inequality (4) implies $p_1 \geq v_{2,1} + v'_{1,2} > v_{2,1} + v_{2,2} - \epsilon > b_1$, which contradicts the budget constraint. Therefore, $x_{1,2} < 1$, and hence, Lemma 4.4 implies that $p_1 = b_1$.

To (b): Lemma 4.4 and (a) show that the utility of agent 1 is $v_{1,1} + x_{1,2}v_{1,2} - b_1$. We select a sufficiently small $\epsilon > 0$, such that $v_{2,1} + v_{2,2} - \epsilon > b_1$ and consider the case where $v'_{1,2} = v_{2,2} - \epsilon$ and all other valuations are unchanged. Lemma 4.3 implies that the utility of agent 1 is $v_{1,1} - v_{2,1}$ in this case. Hence, IC implies that

$$v_{1,1} - v_{2,1} \geq v_{1,1} + x_{1,2}v'_{1,2} - b_1, \qquad \text{and} \tag{5}$$

$$v_{1,1} + x_{1,2}v_{1,2} - b_1 \geq v_{1,1} - v_{2,1}. \tag{6}$$

Inequality (5) implies that $\frac{b_1 - v_{2,1}}{v_{2,2}-\epsilon} = \frac{b_1 - v_{2,1}}{v'_{1,2}} \geq x_{1,2}$. Since this inequality has to hold for all sufficiently small $\epsilon > 0$, we know that $\frac{b_1 - v_{2,1}}{v_{2,2}} \geq x_{1,2}$. Inequality (6) implies that $\frac{b_1 - v_{2,1}}{v_{1,2}} \leq x_{1,2}$.

To (c): Let us assume by contradiction that the inequality $\frac{b_1 - v_{2,1}}{v_{2,2}} \geq x_{1,2}$ implied by (b) is strict, and $\gamma > 0$ is defined such that $\frac{b_1 - v_{2,1}}{v_{2,2}} = x_{1,2} + \gamma$. We select arbitrary $\epsilon > 0$ and $\delta$ with $v_{2,2}\left(\frac{b_1 - v_{2,1}}{b_1 - v_{2,1} - \gamma v_{2,2}} - 1\right) > \delta > 0$ which fulfill $v_{1,2} - \epsilon - \delta = v_{2,2}$. Such variables $\epsilon$ and $\delta$ exist because of $v_{1,2} > v_{2,2}$, and since $v_{1,1} > v_{2,1}$, $b_1 > v_{1,1}$ and $\gamma > 0$ imply that $\frac{b_1 - v_{2,1}}{b_1 - v_{2,1} - \gamma v_{2,2}} > 1$. We consider the alternative case where $v'_{1,2} = v_{1,2} - \epsilon$ and all other valuations are unchanged. We use $x'_{1,2}$ for the fraction of item 2 assigned to agent 1 in this case. By (b) it follows that $\frac{b_1 - v_{2,1}}{v'_{1,2}} \leq x'_{1,2}$, and hence, $\frac{b_1 - v_{2,1}}{v_{2,2}+\delta} \leq x'_{1,2}$. Furthermore, Lemma 4.4 and (a) imply that $p_1 = b_1$ and $x_{1,1} = 1$ in both cases. Now, IC requires that $v_{1,1} + x_{1,2}v_{1,2} - b_1 \geq v_{1,1} + x'_{1,2}v_{1,2} - b_1$, respectively $x_{1,2} \geq x'_{1,2}$, and therefore, $\frac{b_1 - v_{2,1}}{v_{2,2}} - \gamma \geq \frac{b_1 - v_{2,1}}{v_{2,2}+\delta}$. But this inequality can be transformed to $\delta \geq v_{2,2}\left(\frac{b_1 - v_{2,1}}{b_1 - v_{2,1} - \gamma v_{2,2}} - 1\right)$. Contradiction!

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