Valuation Compressions in VCG-Based Combinatorial Auctions

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Abstract. The focus of classic mechanism design has been on truthful direct-revelation mechanisms. In the context of combinatorial auctions the truthful direct-revelation mechanism that maximizes social welfare is the VCG mechanism. For many valuation spaces computing the allocation and payments of the VCG mechanism, however, is a computationally hard problem. We thus study the performance of the VCG mechanism when bidders are forced to choose bids from a subspace of the valuation space for which the VCG outcome can be computed efficiently. We prove improved upper bounds on the welfare loss for restrictions to additive bids and upper and lower bounds for restrictions to non-additive bids. These bounds show that the welfare loss increases in expressiveness. All our bounds apply to equilibrium concepts that can be computed in polynomial time as well as to learning outcomes.

1 Introduction

An important field at the intersection of economics and computer science is the field of mechanism design. The goal of mechanism design is to devise mechanisms consisting of an outcome rule and a payment rule that implement desirable outcomes in strategic equilibrium. A fundamental result in mechanism design theory, the so-called *revelation principle*, asserts that any equilibrium outcome of any mechanism can be obtained as a truthful equilibrium of a direct-revelation mechanism. However, the revelation principle says nothing about the computational complexity of such a truthful direct-revelation mechanism.

In the context of combinatorial auctions the truthful direct-revelation mechanism that maximizes welfare is the *Vickrey-Clarke-Groves (VCG) mechanism* [29, 4, 10]. Unfortunately, for many valuation spaces computing the VCG allocation and payments is a computationally hard problem. This is, for example, the

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case for subadditive, fractionally subadditive, and submodular valuations [16]. We thus study the performance of the VCG mechanism in settings in which the bidders are forced to use bids from a subspace of the valuation space for which the allocation and payments can be computed efficiently. This is obviously the case for additive bids, where the VCG-based mechanism can be interpreted as a separate second-price auction for each item. But it is also the case for the syntactically defined bidding space OXS, which stands for ORs of XORs of singletons, and the semantically defined bidding space GS, which stands for gross substitutes. For OXS bids polynomial-time algorithms for finding a maximum weight matching in a bipartite graph such as the algorithms of [28] and [8] can be used. For GS bids there is a fully polynomial-time approximation scheme due to [15] and polynomial-time algorithms based on linear programming [30] and convolutions of $M^{\#}$-concave functions [21, 20, 22].

One consequence of restrictions of this kind, that we refer to as *valuation compressions*, is that there is typically no longer a truthful dominant-strategy equilibrium that maximizes welfare. We therefore analyze the *Price of Anarchy*, i.e., the ratio between the optimal welfare and the worst possible welfare at equilibrium. We focus on equilibrium concepts such as correlated equilibria and coarse correlated equilibria, which can be computed in polynomial time [24, 13], and naturally emerge from learning processes in which the bidders minimize external or internal regret [7, 11, 17, 2].

Our Contribution. We start our analysis by showing that for restrictions from subadditive valuations to additive bids deciding whether a pure Nash equilibrium exists is $\mathcal{NP}$-hard. This shows the necessity to study other bidding functions or other equilibrium concepts.

We then define a smoothness notion for mechanisms that we refer to as *relaxed smoothness*. This smoothness notion is weaker in some aspects and stronger in another aspect than the weak smoothness notion of [27]. It is weaker in that it allows an agent's deviating bid to depend on the distribution of the bids of the other agents. It is stronger in that it disallows the agent's deviating bid to depend on his own bid. The former gives us more power to choose the deviating bid, and thus has the potential to lead to better bounds. The latter is needed to ensure that the bounds on the welfare loss extend to coarse correlated equilibria and minimization of external regret.

We use relaxed smoothness to prove an upper bound of 4 on the Price of Anarchy with respect to correlated and coarse correlated equilibria. Similarly, we show that the average welfare obtained by minimization of internal and external regret converges to 1/4-th of the optimal welfare. The proofs of these bounds are based on an argument similar to the one in [6]. Our bounds improve the previously known bounds for these solution concepts by a logarithmic factor. We also use relaxed smoothness to prove bounds for restrictions to non-additive bids. For subadditive valuations the bounds are $O(\log(m))$ resp. $\Omega(1/\log(m))$, where m denotes the number of items. For fractionally subadditive valuations the bounds are 2 resp. 1/2. The proofs require novel techniques as non-additive bids lead to non-additive prices for which most of the techniques developed in prior

Table 1. Summary of our results (bold) and the related work (regular) for coarse correlated equilibria and minimization of external regret through repeated play. The range indicates upper and lower bounds on the Price of Anarchy.

		valuations	
		less general	subadditive
bids	additive	[2,2]	[2,**4**]
	more general	[**2**, **2**]	[**2.4**,$\mathbf{O(\log(m))}$]

work fail. The bounds extend the corresponding bounds of [3, 1] from additive to non-additive bids.

Finally, we prove lower bounds on the Price of Anarchy. By showing that VCG-based mechanisms satisfy the *outcome closure property* of [19] we show that the Price of Anarchy with respect to pure Nash equilibria weakly increases with expressiveness. We thus extend the lower bound of 2 from [3] from additive to non-additive bids. This shows that our upper bounds for fractionally subadditive valuations are tight. We prove a lower bound of 2.4 on the Price of Anarchy with respect to pure Nash equilibria that applies to restrictions from subadditive valuations to OXS bids. Together with the upper bound of 2 of [1] for restrictions from subadditive valuations to additive bids this shows that the welfare loss can strictly increase with expressiveness.

Our analysis leaves a number of interesting open questions, both regarding the computation of equilibria and regarding improved upper and lower bounds. Interesting questions regarding the computation of equilibria include whether or not mixed Nash equilibria can be computed efficiently for restrictions from subadditive to additive bids or whether pure Nash equilibria can be computed efficiently for restrictions from fractionally subadditive valuations to additive bids. A particularly interesting open problem regarding improved bounds is whether the welfare loss for computable equilibrium concepts and learning outcomes can be shown to be strictly larger for restrictions to non-additive, say OXS, bids than for restrictions to additive bids. This would show that additive bids are not only sufficient for the best possible bound but also necessary.

Related Work. The Price of Anarchy of restrictions to additive bids is analyzed in [3, 1, 6] for second-price auctions and in [12, 6] for first price auctions. The case where all items are identical, but additional items contribute less to the valuation and agents are forced to place additive bids is analyzed in [18, 14]. Smooth games are defined and analyzed in [25, 26]. The smoothness concept is extended to mechanisms in [27].

Organization. We describe our model in Section 2. We give the hardness result in Section 3, and define relaxed smoothness in Section 4. The upper and lower bounds can be found in Sections 5 to 8. All proofs omitted from this extended abstract are given in the full version of the paper.

2 Preliminaries

Combinatorial Auctions. In a *combinatorial auction* there is a set N of n *agents* and a set M of m *items*. Each agent $i \in N$ employs preferences over bundles of items, represented by a *valuation function* $v_i : 2^M \to \mathbb{R}_{\geq 0}$. We use V_i for the *class of valuation functions* of agent i, and $V = \prod_{i \in N} V_i$ for the class of joint valuations. We write $v = (v_i, v_{-i}) \in V$, where v_i denotes agent i's valuation and v_{-i} denotes the valuations of all agents other than i. We assume that the valuation functions are *normalized* and *monotone*, i.e., $v_i(\emptyset) = 0$ and $v_i(S) \leq v_i(T)$ for all $S \subseteq T$.

A mechanism $M = (f, p)$ is defined by an *allocation rule* $f : B \to \mathcal{P}(M)$ and a *payment rule* $p : B \to \mathbb{R}^n_{\geq 0}$, where B is the *class of bidding functions* and $\mathcal{P}(M)$ denotes the *set of allocations* consisting of all possible partitions X of the set of items M into n sets $X_1, \dots, X_n$. As with valuations we write b_i for agent i's bid, and b_{-i} for the bids by the agents other than i. We define the *social welfare* of an *allocation* X as the sum $\mathrm{SW}(X) = \sum_{i \in N} v_i(X_i)$ of the agents' valuations and use $\mathrm{OPT}(v)$ to denote the maximal achievable social welfare. We say that an allocation rule f is *efficient* if for all bids b it chooses the allocation $f(b)$ that maximizes the sum of the agent's bids, i.e., $\sum_{i \in N} b_i(f_i(b)) = \max_{X \in \mathcal{P}(M)} \sum_{i \in N} b_i(X_i)$. We assume *quasi-linear preferences*, i.e., agent i's *utility* under mechanism M given valuations v and bids b is $u_i(b, v_i) = v_i(f_i(b)) - p_i(b)$.

We focus on the *Vickrey-Clarke-Groves (VCG)* mechanism [29, 4, 10]. Define $b_{-i}(S) = \max_{X \in \mathcal{P}(S)} \sum_{j \neq i} b_j(X_j)$ for all $S \subseteq M$. The VCG mechanisms starts from an efficient allocation rule f and computes the payment of each agent i as $p_i(b) = b_{-i}(M) - b_{-i}(M \setminus f_i(b))$. As the payment $p_i(b)$ only depends on the bundle $f_i(b)$ allocated to agent i and the bids b_{-i} of the agents other than i, we also use $p_i(f_i(b), b_{-i})$ to denote agent i's payment.

If the bids are additive then the VCG prices are additive, i.e., for every agent i and every bundle $S \subseteq M$ we have $p_i(S, b_{-i}) = \sum_{j \in S} \max_{k \neq i} b_k(j)$. Furthermore, the set of items that an agent wins in the VCG mechanism are the items for which he has the highest bid, i.e., agent i wins item j against bids b_{-i} if $b_i(j) \geq \max_{k \neq i} b_k(j) = p_i(j)$ (ignoring ties). Many of the complications in this paper come from the fact that these two observations do *not* apply to non-additive bids.

Valuation Compressions. Our main object of study in this paper are *valuation compressions*, i.e., restrictions of the class of bidding functions B to a strict subclass of the class of valuation functions V.[3] Specifically, we consider valuations and bids from the following hierarchy due to [16],

$$\mathrm{OS} \subset \mathrm{OXS} \subset \mathrm{GS} \subset \mathrm{SM} \subset \mathrm{XOS} \subset \mathrm{CF} \ ,$$

where OS stands for additive, GS for gross substitutes, SM for submodular, and CF for subadditive.

The classes OXS and XOS are syntactically defined. Define OR ($\vee$) as $(u \vee w)(S) = \max_{T \subseteq S}(u(T) + w(S \setminus T))$ and XOR ($\otimes$) as $(u \otimes w)(S) = \max(u(S), w(S))$. Define XS as the class of valuations that assign the same value to all bundles that contain a specific item and zero otherwise. Then OXS is the class of valuations that can be described as ORs of XORs of XS valuations and

[3] This definition is consistent with the notion of *simplification* in [19, 5].

XOS is the class of valuations that can be described by XORs of ORs of XS valuations.

Another important class is the class β-XOS, where $\beta \geq 1$, of β-fractionally subadditive valuations. A valuation v_i is β-fractionally subadditive if for every subset of items T there exists an additive valuation a_i such that (a) $\sum_{j \in T} a_i(j) \geq v_i(T)/\beta$ and (b) $\sum_{j \in S} a_i(j) \leq v_i(S)$ for all $S \subseteq T$. It can be shown that the special case $\beta = 1$ corresponds to the class XOS, and that the class CF is contained in $O(\log(m))$-XOS (see, e.g., Theorem 5.2 in [1]). Functions in XOS are called *fractionally subadditive*.

Solution Concepts. We use game-theoretic reasoning to analyze how agents interact with the mechanism, a desirable criterion being stability according to some solution concept. In the *complete information* model the agents are assumed to know each others' valuations, and in the *incomplete information* model the agents' only know from which distribution the valuations of the other agents are drawn. In the remainder we focus on complete information. The definitions and our results for incomplete information are given in the full version of the paper.

The static solution concepts that we consider in the complete information setting are:

$$\text{DSE} \subset \text{PNE} \subset \text{MNE} \subset \text{CE} \subset \text{CCE} \ ,$$

where DSE stands for dominant strategy equilibrium, PNE for pure Nash equilibrium, MNE for mixed Nash equilibrium, CE for correlated equilibrium, and CCE for coarse correlated equilibrium.

In our analysis we only need the definitions of pure Nash and coarse correlated equilibria. Bids $b \in B$ constitute a *pure Nash equilibrium (PNE)* for valuations $v \in V$ if for every agent $i \in N$ and every bid $b'_i \in B_i$, $u_i(b_i, b_{-i}, v_i) \geq u_i(b'_i, b_{-i}, v_i)$. A distribution $\mathcal{B}$ over bids $b \in B$ is a *coarse correlated equilibrium (CCE)* for valuations $v \in V$ if for every agent $i \in N$ and every pure deviation $b'_i \in B_i$, $\mathrm{E}_{b\sim\mathcal{B}}[u_i(b_i, b_{-i}, v_i)] \geq \mathrm{E}_{b\sim\mathcal{B}}[u_i(b'_i, b_{-i}, v_i)]$.

The dynamic solution concept that we consider in this setting is regret minimization. A sequence of bids $b^1, \dots, b^T$ incurs *vanishing average external regret* if for all agents i, $\sum_{t=1}^{T} u_i(b_i^t, b_{-i}^t, v_i) \geq \max_{b'_i} \sum_{t=1}^{T} u_i(b'_i, b_{-i}^t, v_i) - o(T)$ holds, where $o(\cdot)$ denotes the little-oh notation. The empirical distribution of bids in a sequence of bids that incurs vanishing external regret converges to a coarse correlated equilibrium (see, e.g., Chapter 4 of [23]).

Price of Anarchy. We quantify the welfare loss from valuation compressions by means of the *Price of Anarchy (PoA)*.

The PoA with respect to PNE for valuations $v \in V$ is defined as the worst ratio between the optimal social welfare $\mathrm{OPT}(v)$ and the welfare $\mathrm{SW}(b)$ of a PNE $b \in B$,

$$\mathrm{PoA}(v) = \max_{b\text{: PNE}} \frac{\mathrm{OPT}(v)}{\mathrm{SW}(b)} \ .$$

Similarly, the PoA with respect to MNE, CE, and CCE for valuations $v \in V$ is the worst ratio between the optimal social welfare $\mathrm{SW}(b)$ and the expected welfare $\mathrm{E}_{b\sim\mathcal{B}}[\mathrm{SW}(b)]$ of a MNE, CE, or CCE $\mathcal{B}$,

$$\mathrm{PoA}(v) = \max_{\mathcal{B}\text{: MNE, CE or CCE}} \frac{\mathrm{OPT}(v)}{\mathrm{E}_{b\sim\mathcal{B}}[\mathrm{SW}(b)]} \ .$$

We require that the bids b_i for a given valuation v_i are *conservative*, i.e., $b_i(S) \leq v_i(S)$ for all bundles $S \subseteq M$. Similar assumptions are made and economically justified in the related work [3, 1, 6].

3 Hardness Result for PNE with Additive Bids

Our first result is that deciding whether there exists a pure Nash equilibrium for restrictions from subadditive valuations to additive bids is $\mathcal{NP}$-hard. The proof of this result is by reduction from 3-Partition [9] and uses an example with no pure Nash equilibrium from [1]. The same decision problem is simple for $V \subseteq$ XOS because pure Nash equilibria are guaranteed to exist [3].

Theorem 1. *Suppose that V = CF, B = OS, that the VCG mechanism is used, and that agents bid conservatively. Then it is $\mathcal{NP}$-hard to decide whether there exists a PNE.*

4 Smoothness Notion and Extension Results

Next we define a smoothness notion for mechanisms. It is weaker in some aspects and stronger in another aspect than the weak smoothness notion in [27]. It is weaker because it allows agent i's deviating bid a_i to depend on the marginal distribution $\mathcal{B}_{-i}$ of the bids b_{-i} of the agents other than i. This gives us more power in choosing the deviating bid, which might lead to better bounds. It is stronger because it does *not* allow agent i's deviating bid a_i to depend on his own bid b_i. This allows us to prove bounds that extend to coarse correlated equilibria and not just correlated equilibria.

Definition 1. *A mechanism is relaxed (λ, μ_1, μ_2)-smooth for $\lambda, \mu_1, \mu_2 \geq 0$ if for every valuation profile $v \in V$, every distribution over bids $\mathcal{B}$, and every agent i there exists a bid $a_i(v, \mathcal{B}_{-i})$ such that*

$$\sum_{i\in N} \mathop{\mathrm{E}}_{\mathcal{B}_{-i}}[u_i((a_i, b_{-i}), v_i)] \geq \lambda OPT(v) - \mu_1 \sum_{i\in N} \mathop{\mathrm{E}}_{\mathcal{B}}[p_i(X_i(b), b_{-i})] - \mu_2 \sum_{i\in N} \mathop{\mathrm{E}}_{\mathcal{B}}[b_i(X_i(b))].$$

Theorem 2. *If a mechanism is relaxed (λ, μ_1, μ_2)-smooth, then the Price of Anarchy under conservative bidding with respect to coarse correlated equilibria is at most* $(\max\{\mu_1, 1\} + \mu_2)/\lambda$.

Proof. Fix valuations v. Consider a coarse correlated equilibrium $\mathcal{B}$. For each b from the support of $\mathcal{B}$ denote the allocation for b by $X(b) = (X_1(b), \ldots, X_n(b))$. Let $a = (a_1, \ldots, a_n)$ be defined as in Definition 1. Then,

$$\mathop{\mathrm{E}}_{b\sim\mathcal{B}}[\mathrm{SW}(b)] = \sum_{i\in N} \mathop{\mathrm{E}}_{b\sim\mathcal{B}}[u_i(b, v_i)] + \sum_{i\in N} \mathop{\mathrm{E}}_{b\sim\mathcal{B}}[p_i(X_i(b), b_{-i})]$$

$$\geq \sum_{i \in N} \mathop{\mathrm{E}}_{b_{-i} \sim \mathcal{B}_{-i}}[u_i((a_i, b_{-i}), v_i)] + \sum_{i \in N} \mathop{\mathrm{E}}_{b \sim \mathcal{B}}[p_i(X_i(b), b_{-i})]$$
$$\geq \lambda \ \mathrm{OPT}(v) - (\mu_1 - 1) \sum_{i \in N} \mathop{\mathrm{E}}_{b \sim \mathcal{B}}[p_i(X_i(b), b_{-i})] - \mu_2 \sum_{i \in N} \mathop{\mathrm{E}}_{b \sim \mathcal{B}}[b_i(X_i(b))],$$

where the first equality uses the definition of $u_i(b, v_i)$ as the difference between $v_i(X_i(b))$ and $p_i(X_i(b), b_{-i})$, the first inequality uses the fact that $\mathcal{B}$ is a coarse correlated equilibrium, and the second inequality holds because $a = (a_1, \dots, a_n)$ is defined as in Definition 1.

Since the bids are conservative this can be rearranged to give

$$(1 + \mu_2) \mathop{\mathrm{E}}_{b \sim \mathcal{B}}[\mathrm{SW}(b)] \geq \lambda \ \mathrm{OPT}(v) - (\mu_1 - 1) \sum_{i \in N} \mathop{\mathrm{E}}_{b \sim \mathcal{B}}[p_i(X_i(b), b_{-i})].$$

For $\mu_1 \leq 1$ the second term on the right hand side is lower bounded by zero and the result follows by rearranging terms. For $\mu_1 > 1$ we use that $\mathrm{E}_{b \sim \mathcal{B}}[p_i(X_i(b), b_{-i})] \leq \mathrm{E}_{b \sim \mathcal{B}}[v_i(X_i(b))]$ to lower bound the second term on the right hand side and the result follows by rearranging terms. □

Theorem 3. *If a mechanism is relaxed (λ, μ_1, μ_2)-smooth and $(b^1, \dots, b^T)$ is a sequence of conservative bids with vanishing external regret, then*

$$\frac{1}{T} \sum_{t=1}^{T} SW(b^t) \geq \frac{\lambda}{\max\{\mu_1, 1\} + \mu_2} \cdot OPT(v) - o(1).$$

Proof. Fix valuations v. Consider a sequence of bids $b^1, \dots, b^T$ with vanishing average external regret. For each b^t in the sequence of bids denote the corresponding allocation by $X(b^t) = (X_1(b^t), \dots, X_n(b^t))$. Let $\delta_i^t(a_i) = u_i(a_i, b_{-i}^t, v_i) - u_i(b^t, v_i)$ and let $\Delta(a) = \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} \delta_i^t(a_i)$. Let $a = (a_1, \dots, a_n)$ be defined as in Definition 1, where $\mathcal{B}$ is the empirical distribution of bids. Then,

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^{T} \mathrm{SW}(b^t) &= \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} u_i(b_i^t, b_{-i}^t, v_i) + \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} p_i(X_i(b^t), b_{-i}^t) \\ &= \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} u_i(a_i, b_{-i}^t, v_i) + \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} p_i(X_i(b^t), b_{-i}^t) - \Delta(a) \\ &\geq \lambda \ \mathrm{OPT}(v) - (\mu_1 - 1) \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} p_i(X_i(b^t, b_{-i}^t)) \\ &\qquad - \mu_2 \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{n} b_i(X_i(b^t)) - \Delta(a), \end{aligned}$$

where the first equality uses the definition of $u_i(b_i^t, b_{-i}^t, v_i)$ as the difference between $v_i(X_i(b^t))$ and $p_i(X_i(b^t), b_{-i}^t)$, the second equality uses the definition of $\Delta(a)$, and the third inequality holds because $a = (a_1, \dots, a_n)$ is defined as in Definition 1.

Since the bids are conservative this can be rearranged to give

$$(1+\mu_2)\ \frac{1}{T}\sum_{t=1}^{T}\mathrm{SW}(b^t) \geq \lambda\ \mathrm{OPT}(v) - (\mu_1 - 1)\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{n} p_i(X_i(b^t), b_{-i}^t) - \Delta(a).$$

For $\mu_1 \leq 1$ the second term on the right hand side is lower bounded by zero and the result follows by rearranging terms provided that $\Delta(a) = o(1)$. For $\mu_1 > 1$ we use that $\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{n} p_i(X_i(b^t), b_{-i}^t) \leq \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{n} v_i(X_i(b^t))$ to lower bound the second term on the right hand side and the result follows by rearranging terms provided that $\Delta(a) = o(1)$.

The term $\Delta(a)$ is bounded by $o(1)$ because the sequence of bids $b^1, \ldots, b^T$ incurs vanishing average external regret and, thus,

$$\Delta(a) \leq \frac{1}{T}\sum_{i=1}^{n}\left[\max_{b_i'}\sum_{t=1}^{T} u_i(b_i', b_{-i}^t, v_i) - \sum_{t=1}^{T} u_i(b^t, v_i)\right] \leq \frac{1}{T}\sum_{i=1}^{n} o(T). \qquad \square$$

5 Upper Bounds for CCE and Minimization of External Regret for Additive Bids

We conclude our analysis of restrictions to additive bids by showing how the argument of [6] can be adopted to show that for restrictions from $V = CF$ to $B = OS$ the VCG mechanism is relaxed $(1/2, 0, 1)$-smooth. Using Theorem 2 we obtain an upper bound of 4 on the Price of Anarchy with respect to coarse correlated equilibria. Using Theorem 3 we conclude that the average social welfare for sequences of bids with vanishing external regret converges to at least $1/4$ of the optimal social welfare. We thus improve the best known bounds by a logarithmic factor.

Proposition 1. *Suppose that $V = CF$ and that $B = OS$. Then the VCG mechanism is relaxed $(1/2, 0, 1)$-smooth under conservative bidding.*

To prove this result we need two auxiliary lemmata.

Lemma 1. *Suppose that $V = CF$, that $B = OS$, and that the VCG mechanism is used. Then for every agent i, every bundle Q_i, and every distribution $\mathcal{B}_{-i}$ on the bids b_{-i} of the agents other than i there exists a conservative bid a_i such that*

$$\mathop{\mathrm{E}}_{b_{-i}\sim\mathcal{B}_{-i}}[u_i((a_i, b_{-i}), v_i)] \geq \frac{1}{2}\cdot v_i(Q_i) - \mathop{\mathrm{E}}_{b_{-i}\sim\mathcal{B}_{-i}}[p_i(Q_i, b_{-i})]\ .$$

Proof. Consider bids b_{-i} of the agents $-i$. The bids b_{-i} induce a price $p_i(j) = \max_{k\neq i} b_k(j)$ for each item j. Let T be a maximal subset of items from Q_i such that $v_i(T) \leq p_i(T)$. Define the *truncated* prices q_i as follows:

$$q_i(j) = \begin{cases} p_i(j) & \text{for } j \in Q_i \setminus T\text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

The distribution $\mathcal{B}_{-i}$ on the bids b_{-i} induces a distribution $\mathcal{C}_i$ on the prices p_i as well as a distribution $\mathcal{D}_i$ on the truncated prices q_i.

We would like to allow agent i to draw his bid b_i from the distribution $\mathcal{D}_i$ on the truncated prices q_i. For this we need that (1) the truncated prices are additive and that (2) the truncated prices are conservative. The first condition is satisfied because additive bids lead to additive prices. To see that the second condition is satisfied assume by contradiction that for some set $S \subseteq Q_i \setminus T$, $q_i(S) > v_i(S)$. As $p_i(S) = q_i(S)$ it follows that

$$v_i(S \cup T) \leq v_i(S) + v_i(T) \leq p_i(S) + p_i(T) = p_i(S \cup T),$$

which contradicts our definition of T as a maximal subset of Q_i for which $v_i(T) \leq p_i(T)$.

Consider an arbitrary bid b_i from the support of $\mathcal{D}_i$. Let $X_i(b_i, p_i)$ be the set of items won with bid b_i against prices p_i. Let $Y_i(b_i, q_i)$ be the subset of items from Q_i won with bid b_i against the truncated prices q_i. As $p_i(j) = q_i(j)$ for $j \in Q_i \setminus T$ and $p_i(j) \geq q_i(j)$ for $j \in T$ we have $Y_i(b_i, q_i) \subseteq X_i(b_i, p_i) \cup T$. Thus, using the fact that v_i is subadditive, $v_i(Y_i(b_i, q_i)) \leq v_i(X_i(b_i, p_i)) + v_i(T)$. By the definition of the prices p_i and the truncated prices q_i we have $p_i(Q_i) - q_i(Q_i) = p_i(T) \geq v_i(T)$. By combining these inequalities we obtain

$$v_i(X_i(b_i, p_i)) + p_i(Q_i) \geq v_i(Y_i(b_i, q_i)) + q_i(Q_i).$$

Taking expectations over the prices $p_i \sim \mathcal{C}_i$ and the truncated prices $q_i \sim \mathcal{D}_i$ gives

$$\mathop{\mathrm{E}}_{p_i \sim \mathcal{C}_i}[v_i(X_i(b_i, p_i)) + p_i(Q_i)] \geq \mathop{\mathrm{E}}_{q_i \sim \mathcal{D}_i}[v_i(Y_i(b_i, q_i)) + q_i(Q_i)].$$

Next we take expectations over $b_i \sim \mathcal{D}_i$ on both sides of the inequality. Then we bring the $p_i(Q_i)$ term to the right and the $q_i(Q_i)$ term to the left. Finally, we exploit that the expectation over $q_i \sim \mathcal{D}_i$ of $q_i(Q_i)$ is the same as the expectation over $b_i \sim \mathcal{D}_i$ of $b_i(Q_i)$ to obtain

$$\begin{aligned} &\mathop{\mathrm{E}}_{b_i \sim \mathcal{D}_i}[\mathop{\mathrm{E}}_{p_i \sim \mathcal{C}_i}[v_i(X_i(b_i, p_i))]] - \mathop{\mathrm{E}}_{b_i \sim \mathcal{D}_i}[b_i(Q_i)] \\ &\qquad \geq \mathop{\mathrm{E}}_{b_i \sim \mathcal{D}_i}[\mathop{\mathrm{E}}_{q_i \sim \mathcal{D}_i}[v_i(Y_i(b_i, q_i))]] - \mathop{\mathrm{E}}_{p_i \sim \mathcal{C}_i}[p_i(Q_i)] \end{aligned} \tag{1}$$

Now, using the fact that b_i and q_i are drawn from the same distribution $\mathcal{D}_i$, we can lower bound the first term on the right-hand side of the preceding inequality by

$$\begin{aligned} \mathop{\mathrm{E}}_{b_i \sim \mathcal{D}_i}[\mathop{\mathrm{E}}_{q_i \sim \mathcal{D}_i}[v_i(Y_i(b_i, q_i))]] &= \frac{1}{2} \cdot \mathop{\mathrm{E}}_{b_i \sim \mathcal{D}_i}[\mathop{\mathrm{E}}_{q_i \sim \mathcal{D}_i}[v_i(Y_i(b_i, q_i)) + v_i(Y_i(q_i, b_i))]] \\ &\geq \frac{1}{2} \cdot v_i(Q_i), \end{aligned} \tag{2}$$

where the inequality in the last step comes from the fact that the subset $Y_i(b_i, q_i)$ of Q_i won with bid b_i against prices q_i and the subset $Y_i(q_i, b_i)$ of Q_i won with bid q_i against prices b_i form a partition of Q_i and, thus, because v_i is subadditive, it must be that $v_i(Y_i(b_i, q_i)) + v_i(Y_i(q_i, b_i)) \geq v_i(Q_i)$.

Note that agent i's utility for bid b_i against bids b_{-i} is given by his valuation for the set of items $X_i(b_i, p_i)$ minus the price $p_i(X_i(b_i, p_i))$. Note further that the price $p_i(X_i(b_i, p_i))$ that he faces is at most his bid $b_i(X_i(b_i, p_i))$. Finally note that his bid $b_i(X_i(b_i, p_i))$ is at most $b_i(Q_i)$ because b_i is drawn from $\mathcal{D}_i$. Together with inequality (1) and inequality (2) this shows that

$$\begin{aligned} \mathop{\mathrm{E}}_{b_i \sim \mathcal{D}_i}[\mathop{\mathrm{E}}_{b_{-i} \sim \mathcal{B}_{-i}}[u_i((b_i, b_{-i}), v_i)]] &\geq \mathop{\mathrm{E}}_{b_i \sim \mathcal{D}_i}[\mathop{\mathrm{E}}_{p_i \sim \mathcal{C}_i}[v_i(X_i(b_i, p_i)) - b_i(Q_i)]] \\ &\geq \frac{1}{2} \cdot v_i(Q_i) - \mathop{\mathrm{E}}_{p_i \sim \mathcal{C}_i}[p_i(Q_i)]. \end{aligned}$$

Since this inequality is satisfied in expectation if bid b_i is drawn from distribution $\mathcal{D}_i$ there must be a bid a_i from the support of $\mathcal{D}_i$ that satisfies it. $\square$

Lemma 2. *Suppose that $V = CF$, that $B = OS$, and that the VCG mechanism is used. Then for every partition $Q_1, \dots, Q_n$ of the items and all bids b,*

$$\sum_{i \in N} p_i(Q_i, b_{-i}) \leq \sum_{i \in N} b_i(X_i(b)).$$

Proof. For every agent i and each item $j \in Q_i$ we have $p_i(j, b_{-i}) = \max_{k \neq i} b_k(j) \leq \max_k b_k(j)$. Hence an upper bound on the sum $\sum_{i \in N} p_i(Q_i, b_{-i})$ is given by $\sum_{i \in N} \max_k b_k(j)$. The VCG mechanisms selects allocation $X_1(b), \dots, X_n(b)$ such that $\sum_{i \in N} b_i(X_i(b))$ is maximized. The claim follows. $\square$

Proof of Proposition 1. The claim follows by applying Lemma 1 to every agent i and the corresponding optimal bundle O_i, summing over all agents i, and using Lemma 2 to bound $\mathrm{E}_{b_{-i} \sim \mathcal{B}_{-i}}[\sum_{i \in N} p_i(O_i, b_{-i})]$ by $\mathrm{E}_{b \sim \mathcal{B}}[\sum_{i \in N} b_i(X_i(b))]$. $\square$

An important observation is that the proof of the previous proposition requires that the class of price functions, which is induced by the class of bidding functions via the formula for the VCG payments, is contained in B. While this is the case for additive bids that lead to additive (or "per item") prices this is *not* the case for more expressive bids. In fact, as we will see in the next section, even if the bids are from OXS, the least general class from the hierarchy of [16] that strictly contains the class of additive bids, then the class of price functions that is induced by B is no longer contained in B. This shows that the techniques that led to the results in this section *cannot* be applied to the more expressive bids that we study next.

6 A Lower Bound for PNE with Non-Additive Bids

We start our analysis of non-additive bids with the following separation result: While for restrictions from subadditive valuations to additive bids the bound is 2 for pure Nash equilibria [1], we show that for restrictions from subadditive valuations to OXS bids the corresponding bound is at least 2.4. This shows that more expressiveness can lead to strictly worse bounds.

Theorem 4. *Suppose that $V = CF$, that $OXS \subseteq B \subseteq XOS$, and that the VCG mechanism is used. Then for every $\delta > 0$ there exist valuations v such that the PoA with respect to PNE under conservative bidding is at least $2.4 - \delta$.*

7 Upper Bounds for CCE and Minimization of External Regret for Non-Additive Bids

Our next group of results concerns upper bounds for the PoA for restrictions to non-additive bids. For β-fractionally subadditive valuations we show that the VCG mechanism is relaxed $(1/\beta, 1, 1)$-smooth. By Theorem 2 this implies that the Price of Anarchy with respect to coarse correlated equilibria is at most 2β. By Theorem 3 this implies that the average social welfare obtained in sequences of repeated play with vanishing external regret converges to $1/(2\beta)$ of the optimal social welfare. For subadditive valuations, which are $O(\log(m))$-fractionally subadditive, we thus obtain bounds of $O(\log(m))$ resp. $\Omega(1/\log(m))$. For fractionally subadditive valuations, which are 1-fractionally subadditive, we thus obtain bounds of 2 resp. 1/2. We thus extend the results of [3, 1] from additive to non-additive bids.

Proposition 2. *Suppose that $V \subseteq \beta$-XOS and that $OS \subseteq B \subseteq XOS$, then the VCG mechanism is relaxed $(1/\beta, 1, 1)$-smooth under conservative bidding.*

We will prove that the VCG mechanism satisfies the definition of relaxed smoothness point-wise. For this we need two auxiliary lemmata.

Lemma 3. *Suppose that $V \subseteq \beta$-XOS, that $OS \subseteq B \subseteq XOS$, and that the VCG mechanism is used. Then for all valuations $v \in V$, every agent i, and every bundle of items $Q_i \subseteq M$ there exists a conservative bid $a_i \in B_i$ such that for all conservative bids $b_{-i} \in B_{-i}$, $u_i(a_i, b_{-i}, v_i) \geq \frac{v_i(Q_i)}{\beta} - p_i(Q_i, b_{-i})$.*

Proof. Fix valuations v, agent i, and bundle Q_i. As $v_i \in \beta$-XOS there exists a conservative, additive bid $a_i \in$ OS such that $\sum_{j \in X_i} a_i(j) \leq v_i(X_i)$ for all $X_i \subseteq Q_i$, and $\sum_{j \in Q_i} a_i(j) \geq \frac{v_i(Q_i)}{\beta}$. Consider conservative bids b_{-i}. Suppose that for bids (a_i, b_{-i}) agent i wins items X_i and agents $-i$ win items $M \setminus X_i$. As VCG selects outcome that maximizes the sum of the bids,

$$a_i(X_i) + b_{-i}(M \setminus X_i) \geq a_i(Q_i) + b_{-i}(M \setminus Q_i).$$

We have chosen a_i such that $a_i(X_i) \leq v_i(X_i)$ and $a_i(Q_i) \geq v_i(Q_i)/\beta$. Thus,

$$\begin{aligned} v_i(X_i) + b_{-i}(M \setminus X_i) &\geq a_i(X_i) + b_{-i}(M \setminus X_i) \\ &\geq a_i(Q_i) + b_{-i}(M \setminus Q_i) \geq \frac{v_i(Q_i)}{\beta} + b_{-i}(M \setminus Q_i). \end{aligned}$$

Subtracting $b_{-i}(M)$ from both sides gives

$$v_i(X_i) - p_i(X_i, b_{-i}) \geq \frac{v_i(Q_i)}{\beta} - p_i(Q_i, b_{-i}).$$

As $u_i((a_i, b_{-i}), v_i) = v_i(X_i) - p_i(X_i, b_{-i})$ this shows that $u_i((a_i, b_{-i}), v_i) \geq v_i(Q_i)/\beta - p_i(Q_i, b_{-i})$ as claimed. □

Lemma 4. *Suppose that* $OS \subseteq B \subseteq XOS$ *and that the VCG mechanism is used. For every allocation* $Q_1, \ldots, Q_n$ *and all conservative bids* $b \in B$ *and corresponding allocation* $X_1, \ldots, X_n$, $\sum_{i=1}^n [p_i(Q_i, b_{-i}) - p_i(X_i, b_{-i})] \leq \sum_{i=1}^n b_i(X_i)$.

Proof. We have $p_i(Q_i, b_{-i}) = b_{-i}(M) - b_{-i}(M \setminus Q_i)$ and $p_i(X_i, b_{-i}) = b_{-i}(M) - b_{-i}(M \setminus X_i)$ because the VCG mechanism is used. Thus,

$$\sum_{i=1}^n [p_i(Q_i, b_{-i}) - p_i(X_i, b_{-i})] = \sum_{i=1}^n [b_{-i}(M \setminus X_i) - b_{-i}(M \setminus Q_i)]. \tag{3}$$

We have $b_{-i}(M \setminus X_i) = \sum_{k \neq i} b_k(X_k)$ and $b_{-i}(M \setminus Q_i) \geq \sum_{k \neq i} b_k(X_k \cap (M \setminus Q_i))$ because $(X_k \cap (M \setminus Q_i))_{i \neq k}$ is a feasible allocation of the items $M \setminus Q_i$ among the agents $-i$. Thus,

$$\begin{aligned}
\sum_{i=1}^n [b_{-i}(M \setminus X_i) - b_{-i}(M \setminus Q_i)] &\leq \sum_{i=1}^n [\sum_{k \neq i} b_k(X_k) - \sum_{k \neq i} b_k(X_k \cap (M \setminus Q_i))] \\
&\leq \sum_{i=1}^n [\sum_{k=1}^n b_k(X_k) - \sum_{k=1}^n b_k(X_k \cap (M \setminus Q_i))] \\
&= \sum_{i=1}^n \sum_{k=1}^n b_k(X_k) - \sum_{i=1}^n \sum_{k=1}^n b_k(X_k \cap (M \setminus Q_i)).
\end{aligned} \tag{4}$$

The second inequality holds due to the monotonicity of the bids. Since XOS = 1-XOS for every agent k, bid $b_k \in$ XOS, and set X_k there exists a bid $a_{k,X_k} \in$ OS such that $b_k(X_k) = a_{k,X_k}(X_k) = \sum_{j \in X_k} a_{k,X_k}(j)$ and $b_k(X_k \cap (M \setminus Q_i)) \geq a_{k,X_k}(X_k \cap (M \setminus Q_i)) = \sum_{j \in X_k \cap (M \setminus Q_i)} a_{k,X_k}(j)$ for all i. As $Q_1, \ldots, Q_n$ is a partition of M every item is contained in exactly one of the sets $Q_1, \ldots, Q_n$ and hence in $n-1$ of the sets $M \setminus Q_1, \ldots, M \setminus Q_n$. By the same argument for every agent k and set X_k every item $j \in X_k$ is contained in exactly $n-1$ of the sets $X_k \cap (M \setminus Q_1), \ldots, X_k \cap (M \setminus Q_n)$. Thus, for every fixed k we have that $\sum_{i=1}^n b_k(X_k \cap (M \setminus Q_i)) \geq (n-1) \cdot \sum_{j \in X_k} a_{k,X_k}(j) = (n-1) \cdot a_{k,X_k}(X_k) = (n-1) \cdot b_k(X_k)$. It follows that

$$\begin{aligned}
&\sum_{i=1}^n \sum_{k=1}^n b_k(X_k) - \sum_{i=1}^n \sum_{k=1}^n b_k(X_k \cap (M \setminus Q_i)) \\
&\qquad \leq n \cdot \sum_{k=1}^n b_k(X_k) - (n-1) \cdot \sum_{k=1}^n b_k(X_k) = \sum_{i=1}^n b_k(X_k).
\end{aligned} \tag{5}$$

The claim follows by combining inequalities (3), (4), and (5). □

Proof of Proposition 2. Applying Lemma 3 to the optimal bundles $O_1, \ldots, O_n$ and summing over all agents i,

$$\sum_{i \in N} u_i(a_i, b_{-i}, v) \geq \frac{1}{\beta}\mathrm{OPT}(v) - \sum_{i \in N} p_i(O_i, b_{-i}).$$

Applying Lemma 4 we obtain

$$\sum_{i \in N} u_i(a_i, b_{-i}, v) \geq \frac{1}{\beta}\mathrm{OPT}(v) - \sum_{i \in N} p_i(X_i(b), b_{-i}) - \sum_{i \in N} b_i(X_i(b)). \qquad \square$$

8 More Lower Bounds for PNE with Non-Additive Bids

We conclude by proving matching lower bounds for the VCG mechanism and restrictions from fractionally subadditive valuations to non-additive bids. We prove this result by showing that the VCG mechanism satisfies the *outcome closure property* of [19], which implies that when going from more general bids to less general bids no new pure Nash equilibria are introduced. Hence the lower bound of 2 for pure Nash equilibria and additive bids of [3] translates into a lower bound of 2 for pure Nash equilibria and non-additive bids.

Theorem 5. *Suppose that $OXS \subseteq V \subseteq CF$, that $OS \subseteq B \subseteq XOS$, and that the VCG mechanism is used. Then the PoA with respect to PNE under conservative bidding is at least* 2.

It should be noted that the previous result applies even if valuation and bidding space coincide, and the VCG mechanism has an efficient, dominant-strategy equilibrium. This is because the VCG mechanism also admits other, non-efficient equilibria (and the Price of Anarchy metric does not restrict to dominant-strategy equilibria if they exist).

References

1. Bhawalkar, K., Roughgarden, T.: Welfare guarantees for combinatorial auctions with item bidding. In: Proc. of 22nd SODA. pp. 700–709 (2011)
2. Cesa-Bianchi, N., Mansour, Y., Stoltz, G.: Improved second-order bounds for prediction with expert advice. Machine Learning 66(2-3), 321–352 (2007)
3. Christodoulou, G., Kovács, A., Schapira, M.: Bayesian combinatorial auctions. In: Proc. of 35th ICALP. pp. 820–832 (2008)
4. Clarke, E.H.: Multipart pricing of public goods. Public Choice 11, 17–33 (1971)
5. Dütting, P., Fischer, F., Parkes, D.C.: Simplicity-expressiveness tradeoffs in mechanism design. In: Proc. of 12th EC. pp. 341–350 (2011)
6. Feldman, M., Fu, H., Gravin, N., Lucier, B.: Simultaneous auctions are (almost) efficient. In: Proc. of 45th STOC. pp. 201–210 (2013)
7. Foster, D., Vohra, R.: Calibrated learning and correlated equilibrium. Games and Economic Behavior 21, 40–55 (1997)

8. Fredman, M.L., Tarjan, R.E.: Fibonacci heaps and their uses in improved network optimization algorithms. Journal of the ACM 34(3), 596–615 (1987)
9. Garey, M.R., Johnson, D.S.: Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman and Company, New York (1979)
10. Groves, T.: Incentives in teams. Econometrica 41, 617–631 (1973)
11. Hart, S., Mas-Colell, A.: A simple adaptive procedure leading to correlated equilibrium. Econometrica 68, 1127–1150 (2000)
12. Hassidim, A., Kaplan, H., Mansour, Y., Nisan, N.: Non-price equilibria in markets of discrete goods. In: Proc. of 12th EC. pp. 295–296 (2011)
13. Jiang, A.X., Leyton-Brown, K.: Polynomial-time computation of exact correlated equilibrium in compact games. In: Proc. of 12th EC. pp. 119–126 (2011)
14. de Keijzer, B., Markakis, E., Schäfer, G., Telelis, O.: On the inefficiency of standard multi-unit auctions. http://arxiv.org/abs/1303.1646 pp. 1–16 (2013)
15. Kelso, A.S., Crawford, V.: Job matching, coalition formation, and gross substitutes. Econometrica 50, 1483–1504 (1982)
16. Lehmann, B., Lehmann, D., Nisan, N.: Combinatorial auctions with decreasing marginal utilities. Games and Economic Behavior 55, 270–296 (2005)
17. Littlestone, N., Warmuth, M.K.: The weighted majority algorithm. Information and Computation 108(2), 212–261 (1994)
18. Markakis, E., Telelis, O.: Uniform price auctions: Equilibria and efficiency. In: Proc. of 5th SAGT. pp. 227–238 (2012)
19. Milgrom, P.: Simplified mechanisms with an application to sponsored-search auctions. Games and Economic Behavior 70, 62–70 (2010)
20. Murota, K.: Valuated matroid intersection II: Algorithm. SIAM Journal of Discrete Mathematics 9, 562–576 (1996)
21. Murota, K.: Matrices and Matroids for Systems Analysis. Springer Verlag, Heidelberg (2000)
22. Murota, K., Tamura, A.: Applications of m-convex submodular flow problem to mathematical economics. In: Proc. of 12th ISAAC. pp. 14–25 (2001)
23. Nisan, N., Roughgarden, T., Tardos, E., Vazirani, V.: Algorithmic Game Theory. Cambridge University Press, New York (2007)
24. Papadimitriou, C., Roughgarden, T.: Computing correlated equilibria in multi-player games. Journal of the ACM 55, 14 (2008)
25. Roughgarden, T.: Intrinsic robustness of the price of anarchy. In: Proc. of 41st STOC. pp. 513–522 (2009)
26. Roughgarden, T.: The price of anarchy in games of incomplete information. In: Proc. of 13th EC. pp. 862–879 (2012)
27. Syrgkanis, V., Tardos, É.: Composable and efficient mechanisms. In: Proc. of 45th STOC. pp. 211–220 (2013)
28. Tarjan, R.E.: Data structures and network algorithms. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA (1983)
29. Vickrey, W.: Counterspeculation, auctions, and competitive sealed tenders. J. of Finance 16(1), 8–37 (1961)
30. de Vries, S., Bikhchandani, S., Schummer, J., Vohra, R.V.: Linear programming and Vickrey auctions. In: Dietrich, B., Vohra, R.V. (eds.) Mathematics of the Internet: E-Auctions and Markets, IMA Volumes in Mathematics and its Applications, vol. 127, pp. 75–116. Springer Verlag, Heidelberg (2002)