Prophet Inequalities and Posted Pricing with Samples

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The Classic Prophet Inequality Setup

Generalizing [Krengel and Sucheston 1977, 1978], [Samuel-Cahn 1984]

- $n$ buyers, arriving one by one

- $m$ items

- At each arrival: Decide which items to assign (possibly none)

- Maximize social welfare
The Classic Prophet Inequality Setup

Generalizing [Krengel and Sucheston 1977, 1978], [Samuel-Cahn 1984]

- $n$ buyers, arriving one by one
  - $v_1(\{1\}) = 1$
  - $v_1(\{2\}) = 2$
  - $v_1(\{1, 2\}) = 3$

- $m$ items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare
The Classic Prophet Inequality Setup

- \( n \) buyers, arriving one by one

\[ \begin{align*}
\nu_2(\{1\}) &= 0 \\
\nu_2(\{2\}) &= 10 \\
\nu_2(\{1, 2\}) &= 10
\end{align*} \]

- \( m \) items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare

Generalizing [Krengel and Sucheston 1977, 1978], [Samuel-Cahn 1984]

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$v_2(\{1\}) = 0$
$v_2(\{2\}) = 10$
$v_2(\{1, 2\}) = 10$

0
10
The Classic Prophet Inequality Setup

Generalizing [Krengel and Sucheston 1977, 1978], [Samuel-Cahn 1984]

- *n* buyers, arriving one by one
- *m* items
- At each arrival: Decide which items to assign (possibly none)
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\[ v_1(\{1\}) = 1 \]
\[ v_2(\{1\}) = 1 \]
\[ v_2(\{2\}) = 10 \]
\[ v_2(\{1, 2\}) = 20 \]
\[ v_3(\{1\}) = 5 \]
\[ v_3(\{2\}) = 5 \]
\[ v_3(\{1, 2\}) = 5 \]
The Classic Prophet Inequality Setup

- $n$ buyers, arriving one by one

- $m$ items

- At each arrival: Decide which items to assign (possibly none)

- Maximize social welfare

\[
\begin{align*}
\nu_3(\{1\}) & = 5 \\
\nu_3(\{2\}) & = 5 \\
\nu_3(\{1, 2\}) & = 5
\end{align*}
\]
The Classic Prophet Inequality Setup

Generalizing [Krengel and Sucheston 1977, 1978], [Samuel-Cahn 1984]

- $n$ buyers, arriving one by one
- $m$ items
- At each arrival: Decide which items to assign (possibly none)
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$v_4(\{1\}) = 20$
$v_4(\{2\}) = 50$
$v_4(\{1, 2\}) = 60$
The Classic Prophet Inequality Setup

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v_1(\{1\}) &= 1 \\
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- \( m \) items

- At each arrival: Decide which items to assign (possibly none)

- Maximize social welfare
The Classic Prophet Inequality Setup

Generalizing [Krengel and Sucheston 1977, 1978], [Samuel-Cahn 1984]

- $n$ buyers, arriving one by one

0 10 5 0

- $m$ items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare
- $v_i \sim D_i$ independently; $D_i$ known in advance
Definition

An online algorithm ALG achieves competitive ratio $\alpha \in [0, 1]$ if

$$\inf_{\mathcal{D}} \frac{\mathbb{E}_{v \sim \mathcal{D}}[\text{ALG}(v)]}{\mathbb{E}_{v \sim \mathcal{D}}[\text{OPT}(v)]} \geq \alpha$$

where

- $\text{ALG}(v)$ denotes the social welfare obtained by the online algorithm
- $\text{OPT}(v)$ denotes the optimal social welfare
Connection to Posted Pricing

- \( n \) buyers, arriving one by one

- \( m \) items

- Precompute item prices \( p_1, \ldots, p_m \)

- At each arrival: Arriving buyer purchases bundle maximizing utility \( v_i(S) - \sum_{j \in S} p_j \)

- Maximize social welfare \( \sum_{i=1}^{n} v_i(X_i) \)
Connection to Posted Pricing

- $n$ buyers,
  - $v_1(\{1\}) = 1$
  - $v_1(\{2\}) = 2$
  - $v_1(\{1, 2\}) = 3$

- $m$ items
  - Precompute item prices $p_1, \ldots, p_m$
  - At each arrival: Arriving buyer purchases bundle maximizing utility $v_i(S) - \sum_{j \in S} p_j$
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$v_2(\{1\}) = 0$
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$v_2(\{1, 2\}) = 10$

0
10

4
5

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Connection to Posted Pricing

- n buyers, arriving one by one
- m items
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- At each arrival: Arriving buyer purchases bundle maximizing utility $v_i(S) - \sum_{j \in S} p_j$

- Maximize social welfare $\sum_{i=1}^{n} v_i(X_i)$
Lots of Exciting Work

- Pioneering papers:
  [Hajiaghayi, Kleinberg, Sandholm 2007], [Chawla, Hartline, Malec, Sivan 2010]

- Prophet inequalities for combinatorial settings:
  [Kleinberg and Weinberg 2012], [Gravin, Feldman, Lucier 2015],
  [Feldman, Svensson, Zenklusen 2016], [Dütting, Feldman, Kesselheim, Lucier ’17],
  [Singla and Weinberg 2017], [Gravin and Wang 2019],
  [Dütting, Kesselheim, Lucier 2020]

- Simple mechanisms with near-optimal revenue:
  [Babaioff, Immorlica, Lucier, Weinberg 2014], [Cai and Zhao 2017]

- Prior-free mechanisms:
  [Assadi, Kesselheim, Singla 2021]
Questions & Plan for Today
What if the underlying distributions are unknown?
Questions

What if the underlying distributions are unknown?

What can we do with samples?
Rapidly Growing Literature

- Pioneering paper:
  [Azar, Kleinberg, Weinberg 2014]

- Single-choice problem:
  [Correa, Dütting, Fischer, Schewior 2019], [Rubinstein, Wang, Weinberg 2020],
  [Correa, Cristi, Epstein, Soto 2020], [Kaplan, Naori, Raz 2020]
  [Correa, Dütting, Fischer, Schewior, Ziliotto 2021],
  [Correa, Cristi, Feuilloley, Oosterwijk, Tsigonias-Dimitriadis 2021],
  [Dütting, Lattanzi, Paes Leme, Vassilvitskii 2021],
  [Correa, Cristi, Epstein, Soto 2021+]

- Combinatorial extensions:
  [Caramanis, Dütting, Faw, Fusco, Lazos, Leonard, Papadigenopoulos, Pointourakis,
   Reiffenhäuser 2022],
  [Kaplan, Naori, Raz 2022]
Plan for Today

- Focus on results and techniques for single-choice problem
  - Identical distributions
  - Non-identical distributions
- Briefly discuss combinatorial extensions
Main Techniques: Single-Choice Problem

- Impossibility via connection to extremal combinatorics
- Fresh looking samples
- A card trick (a.k.a. deferred decisions)
- LP-based approach
Main Techniques: Combinatorial Extensions

- Reduction to order-oblivious secretary problem
- Greedy + deferred decisions
1. Single choice, identical distributions
2. Single choice, non-identical distributions
3. Combinatorial extensions
Single Choice, Identical Distributions
The Setting

- One unknown distribution $F$, $k \geq 0$ independent samples from $F$
- Observe $X_1, \ldots, X_n$ drawn independently from $F$, one by one
- Use samples (if any) and $X_1, \ldots, X_j$ to decide on whether or not to stop on $X_j$
- Reward $X_j$ if we stop on $j$
Definition

A stopping rule $r$ is a family of functions $r_1, \ldots, r_n$ where

- $r_i : \mathbb{R}^i \rightarrow [0, 1]$ for all $i$, and
- $r_i(x_1, \ldots, x_i)$ is the probability of stopping at $X_i$ conditioned on having received $X_1 = x_1, \ldots, X_i = x_i$ as values and not having stopped on any of $X_1, \ldots, X_{i-1}$. 
Definition

A \((k, n)\)-stopping rule \(r\) is a family of functions \(r_1, \ldots, r_n\) where

- \(r_i : \mathbb{R}^{k+i} \to [0, 1]\) for all \(i\), and

- \(r_i(s_1, \ldots, s_k, x_1, \ldots, x_i)\) is the probability of stopping at \(X_i\) conditioned on having received \(S_1 = s_1, \ldots, S_k = s_k\) as samples and \(X_1 = x_1, \ldots, X_i = x_i\) as values.

and not having stopped on any of \(X_1, \ldots, X_{i-1}\)
Overview of Results
[Correa, Dütting, Fischer, Schewior 2019]

[0.745, ln(2), 1 - 1/e, 1/e]

[Hill and Kertz 1982; Kertz 1986]
Main Result 1: Without Samples $1/e$ is Best Possible

**Theorem [Correa, Dütting, Fischer, Schewior 2019]**

Let $\delta > 0$. Then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and any $(0, n)$-stopping rule $r$ with stopping time $\tau$ there exists a distribution $F$, not known to the stopping rule, such that when $X_1, \ldots, X_n$ are i.i.d. random variables drawn from $F$,

$$E[X_\tau] \leq \left(\frac{1}{e} + \delta\right) \cdot E\left[\max_{i=1,\ldots,n} X_i\right].$$
Lemma [Correa, Dütting, Fischer, Schewior 2019]

Let $\varepsilon > 0$. There is an infinite subset $S \subseteq \mathbb{N}$ with the following property: For all steps $i$, there exists an $p_i \in [0, 1]$ such that, for all distinct values $v_1, \ldots, v_i \in S$ seen until then:

$$\Pr[r \text{ accepts } v_i \mid v_i = \max \{v_1, \ldots, v_i\}] \in [p_i - \varepsilon, p_i + \varepsilon].$$
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Pr[\text{r accepts } v_1]
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$$

---

complete graph on (infinite set) $S_1$
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complete graph on (infinite set) $S_1$
Establishing the property for $i = 2$

Let $\varepsilon > 0$. There is an infinite subset $S \subseteq \mathbb{N}$ with the following property: For all steps $i$, there exists an $p_i \in [0, 1]$ such that, for all distinct values $v_1, \ldots, v_i \in S$ seen until then:

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![Diagram of a complete graph on (infinite set) $S_1$]
Establishing the property for $i = 2$

Let $\varepsilon > 0$. There is an infinite subset $S \subseteq \mathbb{N}$ with the following property: For all steps $i$, there exists an $p_i \in [0, 1]$ such that, for all distinct values $v_1, \ldots, v_i \in S$ seen until then:

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[Diagram of a complete graph on an infinite set $S_1$.]

Theorem [Ramsey 1930]
There exists a monochromatic infinite induced subgraph.
Let $\varepsilon > 0$. There is an infinite subset $S \subseteq \mathbb{N}$ with the following property: For all steps $i$, there exists an $p_i \in [0, 1]$ such that, for all distinct values $v_1, \ldots, v_i \in S$ seen until then: 
\[ \Pr[r \text{ accepts } v_i \mid v_i = \max\{v_1, \ldots, v_i\}] \in [p_i - \varepsilon, p_i + \varepsilon]. \]

complete graph on (infinite set) $S_1$

**Theorem [Ramsey 1930]**

There exists a monochromatic infinite induced subgraph.
Let $\varepsilon > 0$. There is an infinite subset $S \subseteq \mathbb{N}$ with the following property: For all steps $i$, there exists an $p_i \in [0, 1]$ such that, for all distinct values $v_1, \ldots, v_i \in S$ seen until then:

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**Theorem [Ramsey 1930]**

There exists a monochromatic infinite induced subgraph.
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$$\Pr[r \text{ accepts } v_i \mid v_i = \max\{v_1, \ldots, v_i\}] \in [p_i - \varepsilon, p_i + \varepsilon].$$

**Theorem [Ramsey 1930]**

There exists a monochromatic infinite induced subgraph.
Main Result 2: Can Get $1 - \frac{1}{e}$ With $n - 1$ Samples

Theorem [Correa, Dütting, Fischer, Schewior 2019]

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from an unknown distribution $F$. Then there exists an $(n - 1, n)$-stopping-rule $r$ with stopping time $\tau$ such that

$$
E[X_\tau] \geq \left(1 - \frac{1}{e}\right) \cdot E[\max\{X_1, \ldots, X_n\}].
$$
Warm-Up: Achieving $1 - 1/e$ with $O(n^2)$ Samples

Algorithm:
- Use $n - 1$ fresh samples $S_1^i, \ldots, S_{n-1}^i$ in each step $i$
- Set $\max\{S_1^i, \ldots, S_{n-1}^i\}$ as threshold
Warm-Up: Achieving $1 - 1/e$ with $O(n^2)$ Samples

Algorithm:
- Use $n - 1$ fresh samples $S_1^i, \ldots, S_{n-1}^i$ in each step $i$
- Set $\max\{S_1^i, \ldots, S_{n-1}^i\}$ as threshold

Analysis:

$$E[X_\tau] = \sum_{i=1}^{n} \left(1 - \frac{1}{n}\right)^{i-1} \frac{1}{n} \cdot E[X_i \mid X_i > \max\{S_1^i, \ldots, S_{n-1}^i\}]$$
Warm-Up: Achieving $1 - 1/e$ with $O(n^2)$ Samples

Algorithm:
- Use $n - 1$ fresh samples $S_1^i, \ldots, S_{n-1}^i$ in each step $i$
- Set $\max\{S_1^i, \ldots, S_{n-1}^i\}$ as threshold

Analysis:

$$
E[X_\tau] = \sum_{i=1}^{n} \left( 1 - \frac{1}{n} \right)^{i-1} \frac{1}{n} \cdot \mathbb{E}[X_i \mid X_i > \max\{S_1^i, \ldots, S_{n-1}^i\}] \\
\rightarrow 1 - \frac{1}{e} = \mathbb{E}[\max\{X_1, \ldots, X_n\}]
$$
Idea:

- At each time $i$, select a uniformly random subset $S_i$ of size $n - 1$ from $S_1, \ldots, S_{n-1}, X_1, \ldots, X_{i-1}$.
- Then set $\max S_i$ as a threshold
Idea:

- At each time $i$, select a uniformly random subset $S_i$ of size $n - 1$ from $S_1, \ldots, S_{n-1}, X_1, \ldots, X_{i-1}$.
- Then set $\max S_i$ as a threshold.

**Lemma [Correa, Dütting, Fischer, Schewior 2019]**

Conditioned on arriving in step $i$, the distribution of $S_i$ (as a set) is the one of $n - 1$ fresh samples!
Follow-Up Work

- [Rubinstein, Wang, Weinberg 2020]: Clever adjustment to our construction shows that $O(n)$ samples suffice for 0.745
- [Kaplan, Naor, Raz 2020]: Improved bounds for $k < n$ samples, same $1 - 1/e$ bound for $k = n$ samples
- [Correa et al. 2020]: Two-sided game of Googol, yields improved bound of 0.635 for $k = n$ samples
- [Correa et al. 2021]: Choose sets $S_i$ of varying size, optimal choice yields improved bounds, tight for $k = \beta n$ samples and $\beta \leq 1/(e - 1) \approx 0.58$, bound of 0.649 for $k = n$ samples
- [Correa et al. 2021+]: LP approach to optimal ordinal algorithm; yields 0.671 for $k = n$ samples
Single Choice, Non-Identical Distributions
The Setting

- Now \( n \) unknown distributions \( F_1, \ldots, F_n \), one sample from each distribution
- We get to observe \( X_1, X_2, \ldots \) from \( F_1, F_2, \ldots \) one by one
- Can use samples and \( X_1, \ldots, X_j \) to decide whether (or not) to stop on \( X_j \)
- As before, we get reward \( X_j \) if we stop
An Optimal Single-Sample Prophet Inequality

Theorem [Wang, Rubinstein, Weinberg 2020]*

For any distributions $F_1, \ldots, F_n$ there exists a stopping rule $\tau$ which has access to a single sample from each distribution such that

$$
E[X_\tau] \geq \frac{1}{2} \cdot E\left[\max_{i=1,\ldots,n} X_i\right].
$$

Recall: Factor 1/2 is best possible even with full knowledge of distributions.

* A factor $1/4$ follows from [Azar et al. 2014]
The Algorithm:

- Draw $n$ samples $S_1, \ldots, S_n$ from $F_1, \ldots, F_n$
- Set threshold $T = \max_{i=1,\ldots,n} S_i$
- Accept first $X_i$ such that $X_i \geq T$
Analysis: A Card Trick

- Draw \( 2n \) numbers from \( F_1, \ldots, F_n \); two from each distribution
- Write these pairs of numbers on the two sides of \( n \) cards, so that numbers form the same distribution go to the same card
- Throw the cards in the air (keep order)
- Numbers facing up are samples, numbers facing down are values from which we have to select
Algorithm, Reinterpreted:

- $T =$ largest among numbers that face up
- Turn cards, one by one (in original order)
- Accept first among numbers that were originally facing down that is larger than the threshold
Example

$F_1 \quad F_2 \quad F_3$

First draw: 3 1 6

Second draw: 2 9 7
Example

\[ F_1 \quad F_2 \quad F_3 \]

One side:

\[
\begin{array}{ccc}
3 & 1 & 6 \\
\uparrow & \uparrow & \uparrow \\
2 & 9 & 7 \\
\end{array}
\]
Now throw in the air

\[ F_1 \quad F_2 \quad F_3 \]

\[
\begin{array}{ccc}
2 & 1 & 7 \\
\end{array}
\]

\[ T = 7 \]
Example

\[
\begin{array}{ccc}
F_1 & F_2 & F_3 \\
2 & 1 & 7 \\
\downarrow & & \\
3 & & \\
\end{array}
\]

\[T = 7\]
Example

Flip:

\[ T = 7 \]

\[ F_1 \quad F_2 \quad F_3 \]

2 1 7

\[ \downarrow \quad \downarrow \]

3 9
Lemma [Rubinstein, Wang, Weinberg 2020]

For each realization of the $2n$ numbers, the algorithm that sets the threshold to the maximum of the numbers facing up achieves

$$\mathbb{E}[X_\tau] \geq \frac{1}{2} \cdot \mathbb{E} \left[ \max_{i=1,\ldots,n} X_i \right],$$

where the expectation is over the randomness in cards facing up/down.
Example: The Prophet’s Reward

9 > 7 > 6 > 3 > 2 > 1
Example: The Prophet’s Reward

\[ 9 > 7 > 6 > 3 > 2 > 1 \]

\[ \Pr[\max_i X_i = 9] = \]
Example: The Prophet’s Reward

\[
9 > 7 > 6 > 3 > 2 > 1
\]

down

\[\Pr[\max_i X_i = 9] =\]
Example: The Prophet’s Reward

\[ 9 > 7 > 6 > 3 > 2 > 1 \]

down

\[ \Pr[\max_i X_i = 9] = \frac{1}{2} \]
Example: The Prophet’s Reward

\[
\begin{align*}
9 & > 7 > 6 > 3 > 2 > 1 \\
\text{down}
\end{align*}
\]

\[
\Pr[\max_i X_i = 9] = \frac{1}{2}
\]

\[
\Pr[\max_i X_i = 7] =
\]
Example: The Prophet’s Reward

\[
9 > 7 > 6 > 3 > 2 > 1
\]

down
up down

\[
\Pr[\max_i X_i = 9] = \frac{1}{2}
\]

\[
\Pr[\max_i X_i = 7] =
\]
Example: The Prophet’s Reward

\[
\begin{array}{ccccccc}
9 & > & 7 & > & 6 & > & 3 & > & 2 & > & 1 \\
\text{down} & & & & & & & \\
\text{up} & & & & & & & \\
\text{down} & & & & & & \\
\end{array}
\]

\[
\Pr[\max_i X_i = 9] = \frac{1}{2}
\]

\[
\Pr[\max_i X_i = 7] = \frac{1}{2} \cdot \frac{1}{2}
\]
Example: The Prophet’s Reward

\[9 > 7 > 6 > 3 > 2 > 1\]

down

up down

\[\Pr[\max_i X_i = 9] = \frac{1}{2}\]
\[\Pr[\max_i X_i = 7] = \frac{1}{2} \cdot \frac{1}{2}\]
\[\Pr[\max_i X_i = 6] = \]
Example: The Prophet’s Reward

\[
9 > 7 > 6 > 3 > 2 > 1
\]

\[
\text{down}
\]

\[
\text{up} \quad \text{down}
\]

\[
\text{up} \quad \text{up} \quad \text{down}
\]

\[
\Pr[\max_i X_i = 9] = \frac{1}{2}
\]

\[
\Pr[\max_i X_i = 7] = \frac{1}{2} \cdot \frac{1}{2}
\]

\[
\Pr[\max_i X_i = 6] =
\]
Example: The Prophet’s Reward

\[ 9 > 7 > 6 > 3 > 2 > 1 \]

down
up down
up up down

\[
\Pr[\max_i X_i = 9] = \frac{1}{2}
\]
\[
\Pr[\max_i X_i = 7] = \frac{1}{2} \cdot \frac{1}{2}
\]
\[
\Pr[\max_i X_i = 6] = \frac{1}{2} \cdot \frac{1}{2}
\]
Example: The Prophet’s Reward

pivotal index $j^*$

\[
9 > 7 > 6 > 3 > 2 > 1
\]

down

up  down

up  up  down

\[
\Pr[\max_i X_i = 9] = \frac{1}{2}
\]

\[
\Pr[\max_i X_i = 7] = \frac{1}{2} \cdot \frac{1}{2}
\]

\[
\Pr[\max_i X_i = 6] = \frac{1}{2} \cdot \frac{1}{2}
\]
Claim 1 [Rubinstein et al. 2020]

For any realization \( W_1 > W_2 > \ldots > W_{2n} \) of the \( 2n \) numbers the prophet’s expected reward is

\[
\text{OPT} = \left( \sum_{j=1}^{j^* - 1} \frac{W_j}{2^j} \right) + \frac{W_{j^*}}{2^{j^*-1}}.
\]
Example: The Gambler’s Reward

pivotal index $j^*$

9 > 7 > 6 > 3 > 2 > 1
Example: The Gambler’s Reward

pivotal index $j^*$

9 > 7 > 6 > 3 > 2 > 1

up
Example: The Gambler’s Reward

Pivotal index $j^*$

9 > 7 > 6 > 3 > 2 > 1

up

$\Pr[(\text{up}, \ldots)] = \frac{1}{2}$
Example: The Gambler’s Reward

pivotal index $j^*$

\[ 9 > 7 > 6 > 3 > 2 > 1 \]

up

\[ \Pr[(\text{up, \ldots})] = \frac{1}{2} \]

We get nothing
Example: The Gambler’s Reward

pivotal index $j^*$

\[
\begin{align*}
9 & > 7 & > & 6 & > & 3 & > & 2 & > & 1 \\
\text{up} & & & & & \text{up} \\
\text{down} & & & & & \text{up}
\end{align*}
\]

\[
\Pr[(\text{up}, \ldots)] = \frac{1}{2}
\]

We get nothing.
Example: The Gambler’s Reward

pivotal index $j^*$

9 > 7 > 6 > 3 > 2 > 1

up

down up

$\Pr[(\text{up}, \ldots)] = \frac{1}{2}$

We get nothing

$\Pr[(\text{down, up}, \ldots)] = \frac{1}{2} \cdot \frac{1}{2}$
Example: The Gambler’s Reward

pivotal index $j^*$

\[
9 > 7 > 6 > 3 > 2 > 1
\]

Pr[(up, ...)] = $\frac{1}{2}$

We get nothing

Pr[(down, up, ...)] = $\frac{1}{2} \cdot \frac{1}{2}$

We get 9
Example: The Gambler’s Reward

pivotal index $j^*$

\[
\begin{align*}
9 & > 7 & > 6 & > 3 & > 2 & > 1 \\
\text{up} & & \text{down} & \text{up} & \text{down} & \text{up}
\end{align*}
\]

\[
\Pr[(\text{up}, \ldots)] = \frac{1}{2} \quad \text{We get nothing}
\]

\[
\Pr[(\text{down, up}, \ldots)] = \frac{1}{2} \cdot \frac{1}{2} \quad \text{We get 9}
\]
Example: The Gambler’s Reward

pivotal index $j^*$

9 > 7 > 6 > 3 > 2 > 1

up

down up

down down up

$\Pr[(\text{up, ...})] = \frac{1}{2}$  \hspace{1cm} \text{We get nothing}

$\Pr[(\text{down, up, ...})] = \frac{1}{2} \cdot \frac{1}{2}$  \hspace{1cm} \text{We get 9}

$\Pr[(\text{down, down, up, ...})] = \frac{1}{2} \cdot \frac{1}{2}$
Example: The Gambler’s Reward

pivotal index $j^*$

$\begin{align*}
9 & > 7 & > & 6 & > & 3 & > & 2 & > & 1 \\
\text{up} & \quad & \text{down} & \quad & \text{up} & \quad & \text{down} & \quad & \text{down} & \quad & \text{up}
\end{align*}$

$\Pr[(\text{up}, \ldots)] = \frac{1}{2}$ \quad We get nothing

$\Pr[(\text{down, up}, \ldots)] = \frac{1}{2} \cdot \frac{1}{2}$ \quad We get 9

$\Pr[(\text{down, down, up}, \ldots)] = \frac{1}{2} \cdot \frac{1}{2}$ \quad We get at least 7
### Example: The Gambler’s Reward

**pivotal index $j^*$**

| 9 | > | 7 | > | 6 | > | 3 | > | 2 | > | 1 |

| up | down | up | down | down | up |

Pr[$\text{up, ...} \text{]} = \frac{1}{2}$  

Pr[$\text{down, up, ...} \text{]} = \frac{1}{2} \cdot \frac{1}{2}$  

Pr[$\text{down, down, up, ...} \text{]} \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$  

Pr[$\text{down, down, up, ...} \text{]} \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

We get nothing

We get 9

We get at least 7

We get at least 6
Claim 2 [Rubinstein et al. 2020]

For any realization $W_1 > W_2 > \ldots > W_{2n}$ of the $2n$ numbers the prophet’s expected reward is

$$ALG \geq \left( \sum_{j=1}^{j^*-2} \frac{W_j}{2^{j+1}} \right) + \frac{W_{j^*-1}}{2^{j^*-1}}.$$
Proof of Main Observation

\[ \text{ALG} \geq \left( \sum_{j=1}^{j^*-2} \frac{W_j}{2^{j+1}} \right) + \frac{W_{j^*-1}}{2^{j^*-1}} \]

\[ \geq \left( \sum_{j=1}^{j^*-2} \frac{W_j}{2^{j+1}} \right) + \frac{W_{j^*-1}}{2^{j^*-1}} + \frac{W_{j^*}}{2^{j^*}} \]

\[ = \left( \sum_{j=1}^{j^*-1} \frac{W_j}{2^{j+1}} \right) + \frac{W_{j^*}}{2^{j^*}} \]

\[ = \frac{1}{2} \cdot \left[ \left( \sum_{j=1}^{j^*-1} \frac{W_j}{2^j} \right) + \frac{W_{j^*}}{2^{j^*-1}} \right] = \frac{1}{2} \cdot \text{OPT} \]

q.e.d.
Key Idea: Deferred Decisions

- Defer the decision whether a number is a value or a sample until we reach its position.
- Upon reaching a position we flip a fair coin.
- Coin flips for \( j < j^* \) are independent, outcome on \( j^* \) is deterministic given previous coin tosses.
Extensions
What about richer combinatorial problems such as matroids, matching, combinatorial auctions?
Two Main Techniques

- Reduction to order-oblivious secretary algorithms
  [Azar, Kleinberg, Weinberg 2014]
  [Kaplan, Naori, Raz 2022]

- Greedy + deferred decisions*
  [Caramanis, Dütting, Faw, Fusco, Lazos, Leonardi, Papadigenopoulos, Pountourakis, Reiffenhäuser 2022]
  [Kaplan, Naori, Raz 2022]

* Important predecessor: [Korula and Pal 2009]
Azar, Kleinberg, Weinberg [2014]

<table>
<thead>
<tr>
<th>Setting</th>
<th>Factor</th>
<th>Samples</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$-uniform matroid</td>
<td>$\left(1 - O\left(\frac{1}{\sqrt{k}}\right)\right)^{-1}$</td>
<td>single</td>
</tr>
<tr>
<td>transversal matroid</td>
<td>6</td>
<td>single</td>
</tr>
<tr>
<td>graphic matroid</td>
<td>8</td>
<td>single</td>
</tr>
<tr>
<td>laminar matroid</td>
<td>$12\sqrt{3}$</td>
<td>single</td>
</tr>
<tr>
<td>degree-$d$ bipartite matching</td>
<td>6.75</td>
<td>$O(d)$</td>
</tr>
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<td>(edge arrivals)</td>
<td></td>
<td></td>
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Caramanis et al. [2022]

<table>
<thead>
<tr>
<th>Setting</th>
<th>Previous work</th>
<th>Our Work</th>
</tr>
</thead>
<tbody>
<tr>
<td>general matching, edge arrival</td>
<td>512</td>
<td>16</td>
</tr>
<tr>
<td>budget-additive CAs</td>
<td>N/A</td>
<td>24</td>
</tr>
<tr>
<td>bipartite matching, edge arrival</td>
<td>256</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td>6.75 (degree-(d))</td>
<td>16 (any degree)</td>
</tr>
<tr>
<td></td>
<td>(O(d^2)) samples</td>
<td>1 sample</td>
</tr>
<tr>
<td>bipartite matching, vertex arrivals</td>
<td>13.5</td>
<td>8</td>
</tr>
<tr>
<td>transversal matroid</td>
<td>16</td>
<td>8</td>
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<tr>
<td>graphic matroid</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>low density matroid</td>
<td>(4\gamma(M)^3)</td>
<td>(2\gamma(M))</td>
</tr>
<tr>
<td>column (k)-sparse linear matroid</td>
<td>(4k)</td>
<td>(2k)</td>
</tr>
<tr>
<td>Setting</td>
<td>Factor</td>
<td></td>
</tr>
<tr>
<td>---------------------------------------------</td>
<td>--------------------</td>
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</tr>
<tr>
<td>bipartite matching, vertex arrivals</td>
<td>$3 - 2\sqrt{2} \approx 5.83$</td>
<td></td>
</tr>
<tr>
<td>general matching, edge arrivals</td>
<td>13.5</td>
<td></td>
</tr>
</tbody>
</table>
Theorem (Informal) [Caramanis et al. [2022]]

There is a constant-factor preserving reduction from order-oblivious secretary to “pointwise” single-sample prophet inequalities.*

* Definition related to deferred decision approach. See paper for formal definition.
Conclusion & Directions
Summary of results and techniques for sample-based prophet inequalities
- Focused on single-choice problem
- Briefly discussed extensions to combinatorial domains

Understand boundaries/relative strength of single-sample prophet inequalities
- $O(1)$ for general matroids?
- $O(1)$ for submodular/XOS combinatorial auctions?

Additional technical innovations?
Conclusion & Directions

- Summary of results and techniques for sample-based prophet inequalities
  - Focused on single-choice problem
  - Briefly discussed extensions to combinatorial domains

- Understand boundaries/relative strength of single-sample prophet inequalities
  - $O(1)$ for general matroids?
  - $O(1)$ for submodular/XOS combinatorial auctions?

- Additional technical innovations?

Thanks! Questions?!