Prophet Inequalities and Posted Pricing with Samples

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The Classic Prophet Inequality Setup

Generalizing [Krengel and Sucheston 1977, 1978], [Samuel-Cahn 1984]

- n buyers, arriving one by one

- m items

- At each arrival: Decide which items to assign (possibly none)
- Maximize social welfare

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$$\begin{aligned} v_1(\{1\}) &= 1 \\ v_1(\{2\}) &= 2 \\ v_1(\{1,2\}) &= 3 \end{aligned}$$

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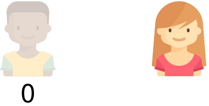

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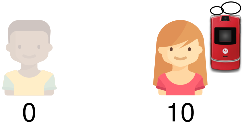

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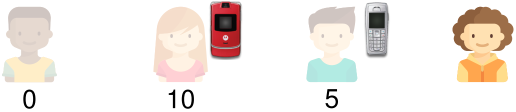

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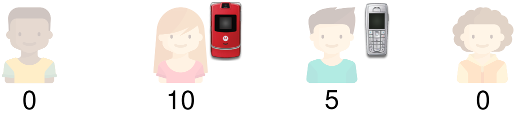

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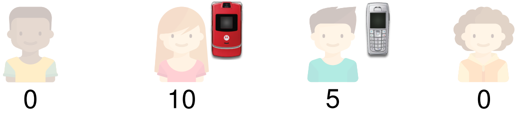

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- Maximize social welfare
- $v_i \sim \mathcal{D}_i$ independently; $\mathcal{D}_i$ known in advance

The Prophet Benchmark

Definition

An online algorithm ALG achieves competitive ratio $\alpha \in [0, 1]$ if

$$\inf_{\mathcal{D}} \frac{\mathbf{E}_{v \sim \mathcal{D}}[\mathrm{ALG}(v)]}{\mathbf{E}_{v \sim \mathcal{D}}[\mathrm{OPT}(v)]} \geq \alpha$$

where

- ALG(v) denotes the social welfare obtained by the online algorithm
- OPT(v) denotes the optimal social welfare

Connection to Posted Pricing

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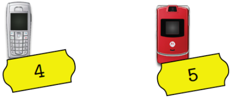

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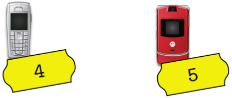

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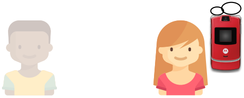

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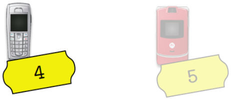

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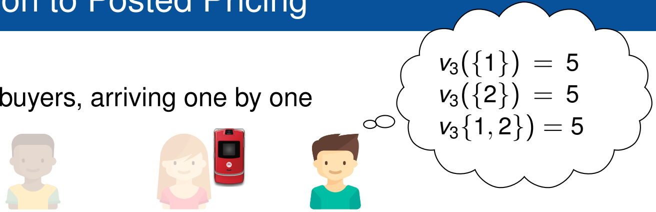

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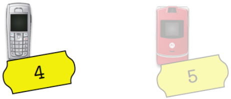

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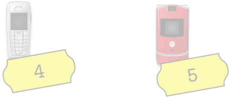

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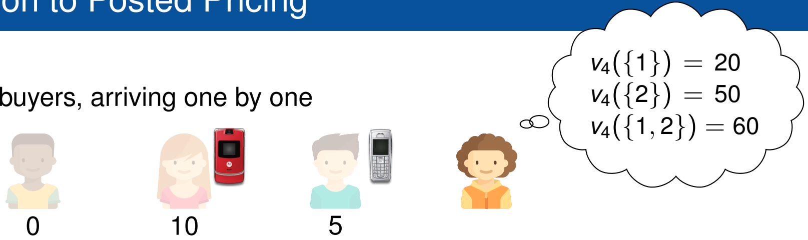

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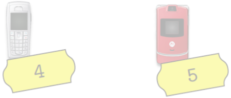

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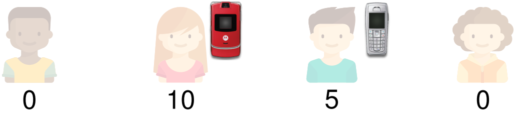

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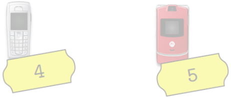

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Lots of Exciting Work

- Pioneering papers:
  [Hajiaghayi, Kleinberg, Sandholm 2007], [Chawla, Hartline, Malec, Sivan 2010]
- Prophet inequalities for combinatorial settings:
  [Kleinberg and Weinberg 2012], [Gravin, Feldman, Lucier 2015],
  [Feldman, Svensson, Zenklusen 2016], [Dütting, Feldman, Kesselheim, Lucier '17],
  [Singla and Weinberg 2017], [Gravin and Wang 2019],
  [Dütting, Kesselheim, Lucier 2020]
- Simple mechanisms with near-optimal revenue:
  [Babaioff, Immorlica, Lucier, Weinberg 2014], [Cai and Zhao 2017]
- Prior-free mechanisms:
  [Assadi, Kesselheim, Singla 2021]

Questions & Plan for Today

Questions

What if the underlying distributions are unknown?

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What can we do with samples?

Rapidly Growing Literature

- Pioneering paper:
  [Azar, Kleinberg, Weinberg 2014]
- Single-choice problem:
  [Correa, Dütting, Fischer, Schewior 2019], [Rubinstein, Wang, Weinberg 2020],
  [Correa, Cristi, Epstein, Soto 2020], [Kaplan, Naori, Raz 2020]
  [Correa, Dütting, Fischer, Schewior, Ziliotto 2021],
  [Correa, Cristi, Feuilloley, Oosterwijk, Tsigonias-Dimitriadis 2021],
  [Dütting, Lattanzi, Paes Leme, Vassilvitskii 2021],
  [Correa, Cristi, Epstein, Soto 2021+]
- Combinatorial extensions:
  [Caramanis, Dütting, Faw, Fusco, Lazos, Leonardi, Papadigenopoulos, Pountourakis, Reiffenhäuser 2022],
  [Kaplan, Naori, Raz 2022]

Plan for Today

- Focus on results and techniques for single-choice problem
    - Identical distributions
    - Non-identical distributions
- Briefly discuss combinatorial extensions

Main Techniques: Single-Choice Problem

- Impossibility via connection to extremal combinatorics
- Fresh looking samples
- A card trick (a.k.a. deferred decisions)
- LP-based approach

Main Techniques: Combinatorial Extensions

- Reduction to order-oblivious secretary problem
- Greedy + deferred decisions

Outline

1. Single choice, identical distributions
2. Single choice, non-identical distributions
3. Combinatorial extensions

Single Choice, Identical Distributions

The Setting

- One unknown distribution F, $k \geq 0$ independent samples from F
- Observe $X_1, \ldots, X_n$ drawn independently from F, one by one
- Use samples (if any) and $X_1, \ldots, X_j$ to decide on whether or not to stop on X_j
- Reward X_j if we stop on j

Stopping Rule

Definition

A stopping rule **r** is a family of functions $r_1, \ldots, r_n$ where

- $r_i : \mathbb{R}^i \rightarrow [0, 1]$ for all i, and
- $r_i(x_1, \ldots, x_i)$ is the probability of stopping at X_i conditioned on having received $X_1 = x_1, \ldots, X_i = x_i$ as values and not having stopped on any of $X_1, \ldots, X_{i-1}$.

(k, n)-Stopping Rule

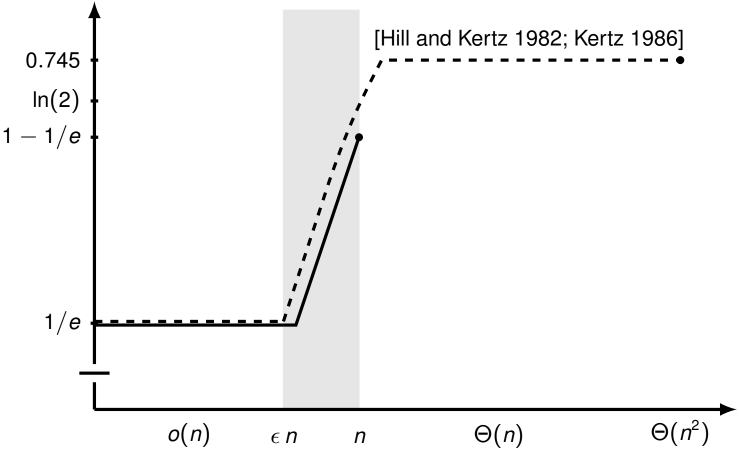
Definition

A (k, n)-stopping rule $\mathbf{r}$ is a family of functions $r_1, \ldots, r_n$ where

- $r_i : \mathbb{R}^{k+i} \rightarrow [0, 1]$ for all i, and
- $r_i(s_1, \ldots, s_k, x_1, \ldots, x_i)$ is the probability of stopping at X_i conditioned on having received $S_1 = s_1, \ldots, S_k = s_k$ as samples and $X_1 = x_1, \ldots, X_i = x_i$ as values. and not having stopped on any of $X_1, \ldots, X_{i-1}$

Overview of Results

[Correa, Dütting, Fischer, Schewior 2019]



Main Result 1: Without Samples $1/e$ is Best Possible

Theorem [Correa, Dütting, Fischer, Schewior 2019]

Let $\delta > 0$.Then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and any $(0, n)$-stopping rule $\mathbf{r}$ with stopping time τ there exists a distribution F, not known to the stopping rule, such that when $X_1, \ldots, X_n$ are i.i.d. random variables drawn from F,

$$\mathbf{E}[X_\tau] \leq \left(\frac{1}{e} + \delta\right) \cdot \mathbf{E}\left[\max_{i=1,\ldots,n} X_i\right].$$

Structural Lemma

Lemma [Correa, Dütting, Fischer, Schewior 2019]

Let $\varepsilon > 0$. There is an infinite subset $S \subseteq \mathbb{N}$ with the following property: For all steps i, there exists an $p_i \in [0, 1]$ such that, for all distinct values $v_1, \ldots, v_i \in S$ seen until then:

$$\Pr[\mathbf{r} \text{ accepts } v_i \mid v_i = \max\{v_1, \ldots, v_i\}] \in [p_i - \varepsilon, p_i + \varepsilon].$$

Establishing the property for $i = 1$

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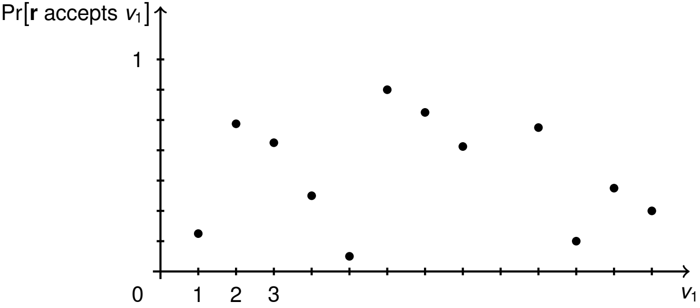

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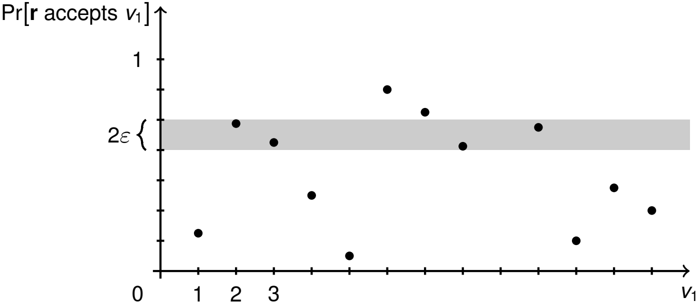

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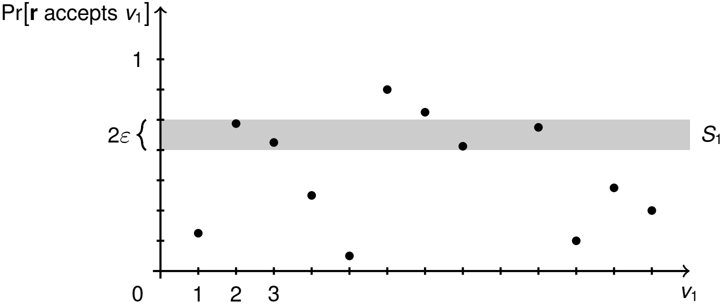

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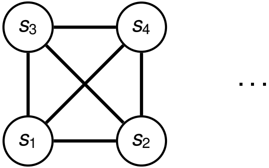

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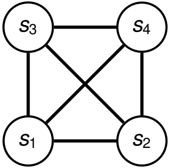

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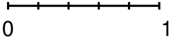

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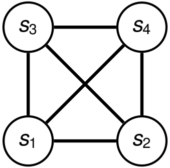

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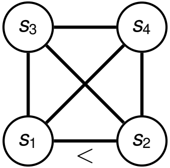

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Theorem [Ramsey 1930]

There exists a monochromatic infinite induced subgraph.

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$$\Pr[\mathbf{r} \text{ accepts } v_i \mid v_i = \max\{v_1, \dots, v_i\}] \in [p_i - \varepsilon, p_i + \varepsilon].$$

complete graph on (infinite set) S_1

Theorem [Ramsey 1930]

There exists a monochromatic infinite induced subgraph.

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complete graph on (infinite set) S_1

Theorem [Ramsey 1930]

There exists a monochromatic infinite induced subgraph.

Main Result 2: Can Get $1 - 1/e$ With $n - 1$ Samples

Theorem [Correa, Dütting, Fischer, Schewior 2019]

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from an unknown distribution F. Then there exists an $(n-1, n)$-stopping-rule $\mathbf{r}$ with stopping time τ such that

$$\mathbf{E}\left[X_\tau\right] \geq \left(1 - \frac{1}{e}\right) \cdot \mathbf{E}\left[\max\{X_1, \ldots, X_n\}\right].$$

Warm-Up: Achieving $1 - 1/e$ with $O(n^2)$ Samples

Algorithm:

- Use $n - 1$ fresh samples $S^i_1, \ldots, S^i_{n-1}$ in each step i
- Set $\max\{S^i_1, \ldots, S^i_{n-1}\}$ as threshold

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- Use $n - 1$ fresh samples $S^i_1, \ldots, S^i_{n-1}$ in each step i
- Set $\max\{S^i_1, \ldots, S^i_{n-1}\}$ as threshold

Analysis:

$$\mathbf{E}[X_\tau] = \sum_{i=1}^{n} \left(1 - \frac{1}{n}\right)^{i-1} \frac{1}{n} \cdot \mathbf{E}[X_i \mid X_i > \max\{S^i_1, \ldots, S^i_{n-1}]\}$$

Warm-Up: Achieving $1 - 1/e$ with $O(n^2)$ Samples

Algorithm:

- Use $n - 1$ fresh samples $S^i_1, \ldots, S^i_{n-1}$ in each step i
- Set $\max\{S^i_1, \ldots, S^i_{n-1}\}$ as threshold

Analysis:

$$\mathbf{E}[X_\tau] = \underbrace{\sum_{i=1}^{n} \left(1 - \frac{1}{n}\right)^{i-1} \frac{1}{n}}_{\to 1 - \frac{1}{e}} \cdot \underbrace{\mathbf{E}[X_i \mid X_i > \max\{S^i_1, \ldots, S^i_{n-1}]\}}_{= \mathbf{E}[\max\{X_1, \ldots, X_n]}$$

Fresh Looking Samples

Idea:

- At each time i, select a uniformly random subset $\mathcal{S}_i$ of size $n - 1$ from $S_1, \ldots, S_{n-1}, X_1, \ldots, X_{i-1}$.
- Then set $\max \mathcal{S}_i$ as a threshold

Fresh Looking Samples

Idea:

- At each time i, select a uniformly random subset $\mathcal{S}_i$ of size $n-1$ from $S_1, \ldots, S_{n-1}, X_1, \ldots, X_{i-1}$.
- Then set $\max \mathcal{S}_i$ as a threshold

Lemma [Correa, Dütting, Fischer, Schewior 2019]

Conditioned on arriving in step i, the distribution of $\mathcal{S}_i$ (as a set) is the one of $n-1$ fresh samples!

Follow-Up Work

- [Rubinstein, Wang, Weinberg 2020]: Clever adjustment to our construction shows that $O(n)$ samples suffice for 0.745
- [Kaplan, Naor, Raz 2020]: Improved bounds for $k < n$ samples, same $1 - 1/e$ bound for $k = n$ samples
- [Correa et al. 2020]: Two-sided game of Googol, yields improved bound of 0.635 for $k = n$ samples
- [Correa et al. 2021]: Choose sets $\mathcal{S}_i$ of varying size, optimal choice yields improved bounds, tight for $k = \beta n$ samples and $\beta \leq 1/(e - 1) \approx 0.58$, bound of 0.649 for $k = n$ samples
- [Correa et al. 2021+]: LP approach to optimal ordinal algorithm; yields 0.671 for $k = n$ samples

Single Choice, Non-Identical Distributions

The Setting

- Now n unknown distributions $F_1, \ldots, F_n$, one sample from each distribution
- We get to observe X_1, X_2, . . . from F_1, F_2, . . . one by one
- Can use samples and $X_1, \ldots, X_j$ to decide whether (or not) to stop on X_j
- As before, we get reward X_j if we stop

An Optimal Single-Sample Prophet Inequality

Theorem [Wang, Rubinstein, Weinberg 2020]*

For any distributions $F_1, \ldots, F_n$ there exists a stopping rule τ which has access to a single sample from each distribution such that

$$\mathbf{E}[X_\tau] \geq \frac{1}{2} \cdot \mathbf{E}\left[\max_{i=1,\ldots,n} X_i\right].$$

Recall: Factor $1/2$ is best possible even with full knowledge of distributions.

* A factor $1/4$ follows from [Azar et al. 2014]

The Optimal Strategy

The Algorithm:

- Draw n samples $S_1, \ldots, S_n$ from $F_1, \ldots, F_n$
- Set threshold $T = \max_{i=1,\ldots,n} S_i$
- Accept first X_i such that $X_i \geq T$

Analysis: A Card Trick

- Draw $2n$ numbers from $F_1, \ldots, F_n$; two from each distribution
- Write these pairs of numbers on the two sides of n cards, so that numbers form the same distribution go to the same card
- Throw the cards in the air (keep order)
- Numbers facing up are samples, numbers facing down are values from which we have to select

Reinterprete Algorithm

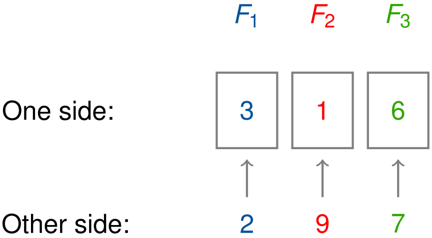
Algorithm, Reinterpreted:

- T = largest among numbers that face up
- Turn cards, one by one (in original order)
- Accept first among numbers that were originally facing down that is larger than the threshold

Example

	F_1	F_2	F_3
First draw:	3	1	6
Second draw:	2	9	7

Example



	F_1	F_2	F_3
One side:	3	1	6
Other side:	2	9	7

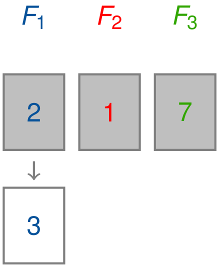
Example

Now throw in the air

F_1	F_2	F_3
2	1	7

$T = 7$

Example



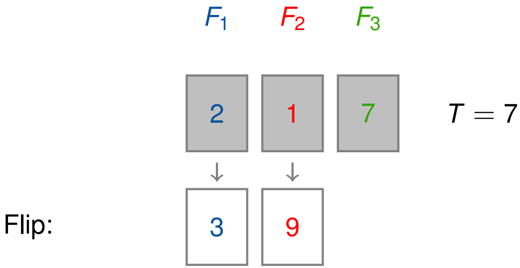
F_1 F_2 F_3

2 1 7

$T = 7$

Flip: 3

Example



F_1 F_2 F_3

2 1 7

$T = 7$

Flip: 3 9

Main Observation

Lemma [Rubinstein, Wang, Weinberg 2020]

For each realization of the $2n$ numbers, the algorithm that sets the threshold to the maximum of the numbers facing up achieves

$$\mathbf{E}[X_\tau] \geq \frac{1}{2} \cdot \mathbf{E}\left[\max_{i=1,\ldots,n} X_i\right],$$

where the expectation is over the randomness in cards facing up/down.

Example: The Prophet’s Reward

$9 > 7 > 6 > 3 > 2 > 1$

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$\mathbf{Pr}[\max_i X_i = 9] =$

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down

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$\mathbf{Pr}[\max_i X_i = 9] = \frac{1}{2}$

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$\mathbf{Pr}[\max_i X_i = 7] =$

Example: The Prophet's Reward

$9 > 7 > 6 > 3 > 2 > 1$

9	7
down	
up	down

$\mathbf{Pr}[\max_i X_i = 9] = \frac{1}{2}$

$\mathbf{Pr}[\max_i X_i = 7] =$

Example: The Prophet's Reward

$9 > 7 > 6 > 3 > 2 > 1$

down

up down

$\mathbf{Pr}[\max_i X_i = 9] = \frac{1}{2}$

$\mathbf{Pr}[\max_i X_i = 7] = \frac{1}{2} \cdot \frac{1}{2}$

Example: The Prophet's Reward

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down

up down

$\mathbf{Pr}[\max_i X_i = 9] = \frac{1}{2}$

$\mathbf{Pr}[\max_i X_i = 7] = \frac{1}{2} \cdot \frac{1}{2}$

$\mathbf{Pr}[\max_i X_i = 6] =$

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down		
up	down	
up	up	down

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Example: The Prophet's Reward

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$$\mathbf{Pr}[\max_i X_i = 6] = \frac{1}{2} \cdot \frac{1}{2}$$

Example: The Prophet's Reward

pivotal index j^*

$9 > 7 > \textcircled{6} > 3 > 2 > 1$

down		
up	down	
up	up	down

$$\mathbf{Pr}[\max_i X_i = 9] = \tfrac{1}{2}$$

$$\mathbf{Pr}[\max_i X_i = 7] = \tfrac{1}{2} \cdot \tfrac{1}{2}$$

$$\mathbf{Pr}[\max_i X_i = 6] = \tfrac{1}{2} \cdot \tfrac{1}{2}$$

The Prophet's Reward

Claim 1 [Rubinstein et al. 2020]

For any realization $W_1 > W_2 > \ldots > W_{2n}$ of the $2n$ numbers the prophet's expected reward is

$$\mathrm{OPT} = \left(\sum_{j=1}^{j^*-1} \frac{W_j}{2^j}\right) + \frac{W_{j*}}{2^{j*-1}}.$$

Example: The Gambler’s Reward

pivotal index j^*

$$9 > 7 > \textcircled{6} > 3 > 2 > 1$$

Example: The Gambler’s Reward

pivotal index j^*

$$9 > 7 > \bigcirc\!\!\!\!\!6 > 3 > 2 > 1$$

up

Example: The Gambler's Reward

pivotal index j^*

$$9 > 7 > \boxed{6} > 3 > 2 > 1$$

up

$\mathbf{Pr}[(\text{up}, \ldots)] = \frac{1}{2}$

Example: The Gambler’s Reward

pivotal index j^*

$9 > 7 > \boxed{6} > 3 > 2 > 1$

up

$\mathbf{Pr}[(\text{up}, \ldots)] = \frac{1}{2}$

We get nothing

Example: The Gambler’s Reward

pivotal index j^*

9	$>$	7	$>$	6	$>$	3	$>$	2	$>$	1
up										
down		up								

$\mathbf{Pr}[(\text{up}, \ldots)] = \frac{1}{2}$

We get nothing

Example: The Gambler's Reward

pivotal index j^*

$$9 > 7 > \enclose{circle}{6} > 3 > 2 > 1$$

up					
down	up				

$\mathbf{Pr}[(\text{up}, \ldots)] = \frac{1}{2}$

We get nothing

$\mathbf{Pr}[(\text{down}, \text{up}, \ldots)] = \frac{1}{2} \cdot \frac{1}{2}$

Example: The Gambler's Reward

pivotal index j^*

$9 > 7 >$ ⑥ $> 3 > 2 > 1$

up

down up

$\textbf{Pr}[(\text{up}, \ldots)] = \frac{1}{2}$ We get nothing

$\textbf{Pr}[(\text{down}, \text{up}, \ldots)] = \frac{1}{2} \cdot \frac{1}{2}$ We get 9

Example: The Gambler’s Reward

pivotal index j^*

9	>	7	>	⑥	>	3	>	2	>	1
up										
down		up								
down		down		up						

$\textbf{Pr}[(\text{up}, \ldots)] = \frac{1}{2}$ We get nothing

$\textbf{Pr}[(\text{down}, \text{up}, \ldots)] = \frac{1}{2} \cdot \frac{1}{2}$ We get 9

Example: The Gambler's Reward

pivotal index j^*

9	$>$	7	$>$	⑥	$>$	3	$>$	2	$>$	1
up										
down		up								
down		down		up						

$\mathbf{Pr}[(\text{up}, \ldots)] = \frac{1}{2}$	We get nothing
$\mathbf{Pr}[(\text{down}, \text{up}, \ldots)] = \frac{1}{2} \cdot \frac{1}{2}$	We get 9
$\mathbf{Pr}[(\text{down}, \text{down}, \text{up}, \ldots)] = \frac{1}{2} \cdot \frac{1}{2}$	

Example: The Gambler's Reward

pivotal index j^*

9 >	7 >	$\textcircled{6}$ >	3 >	2 >	1
up					
down	up				
down	down	up			

$\textbf{Pr}[(\text{up}, \ldots)] = \frac{1}{2}$ We get nothing

$\textbf{Pr}[(\text{down}, \text{up}, \ldots)] = \frac{1}{2} \cdot \frac{1}{2}$ We get 9

$\textbf{Pr}[(\text{down}, \text{down}, \text{up}, \ldots)] = \frac{1}{2} \cdot \frac{1}{2}$ We get at least 7

Example: The Gambler’s Reward

pivotal index j^*

9	>	7	>	⑥	>	3	>	2	>	1
up										
down		up								
down		down		up						

$\mathbf{Pr}[(\text{up}, \ldots)] = \frac{1}{2}$ We get nothing

$\mathbf{Pr}[(\text{down}, \text{up}, \ldots)] = \frac{1}{2} \cdot \frac{1}{2}$ We get 9

$\mathbf{Pr}[(\text{down}, \text{down}, \text{up}, \ldots)] \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$ We get at least 7

$\mathbf{Pr}[(\text{down}, \text{down}, \text{up}, \ldots)] \cdot \frac{1}{2} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$ We get at least 6

The Gambler's Reward

Claim 2 [Rubinstein et al. 2020]

For any realization $W_1 > W_2 > \ldots > W_{2n}$ of the $2n$ numbers the prophet's expected reward is

$$\mathrm{ALG} \geq \left(\sum_{j=1}^{j^*-2} \frac{W_j}{2^{j+1}} \right) + \frac{W_{j*-1}}{2^{j^*-1}}.$$

Proof of Main Observation

$$
\begin{aligned}
\mathrm{ALG} &\geq \left(\sum_{j=1}^{j^*-2} \frac{W_j}{2^{j+1}}\right) + \frac{W_{j*-1}}{2^{j^*-1}} \\
&\geq \left(\sum_{j=1}^{j^*-2} \frac{W_j}{2^{j+1}}\right) + \frac{W_{j*-1}}{2^{j^*}} + \frac{W_{j^*}}{2^{j^*}} \\
&= \left(\sum_{j=1}^{j^*-1} \frac{W_j}{2^{j+1}}\right) + \frac{W_{j^*}}{2^{j^*}} \\
&= \frac{1}{2} \cdot \left[\left(\sum_{j=1}^{j^*-1} \frac{W_j}{2^{j}}\right) + \frac{W_{j^*}}{2^{j^*-1}}\right] = \frac{1}{2} \cdot \mathrm{OPT}
\end{aligned}
$$

q.e.d.

Key Idea: Deferred Decisions

- Defer the decision whether a number is a value or a sample until we reach its position
- Upon reaching a position we flip a fair coin
- Coin flips for $j < j^*$ are independent, outcome on j^* is deterministic given previous coin tosses

Extensions

Question

What about richer combinatorial problems such as matroids, matching, combinatorial auctions?

Two Main Techniques

- Reduction to order-oblivious secretary algorithms
  [Azar, Kleinberg, Weinberg 2014]
  [Kaplan, Naori, Raz 2022]
- Greedy + deferred decisions*
  [Caramanis, Dütting, Faw, Fusco, Lazos, Leonardi, Papadigenopoulos, Pountourakis, Reiffenhäuser 2022]
  [Kaplan, Naori, Raz 2022]

* Important predecessor: [Korula and Pal 2009]

Azar, Kleinberg, Weinberg [2014]

Setting	Factor	Samples
k-uniform matroid	$\left(1 - O\left(\frac{1}{\sqrt{k}}\right)\right)^{-1}$	single
transversal matroid	6	single
graphic matroid	8	single
laminar matroid	$12\sqrt{3}$	single
degree-d bipartite matching (edge arrivals)	6.75	$O(d)$

Caramanis et al. [2022]

Setting	Previous work	Our Work
general matching, edge arrival	512	16
budget-additive CAs	N/A	24
bipartite matching, edge arrival	256	16
	6.75 (degree-d)	16 (any degree)
	$O(d^2)$ samples	1 sample
bipartite matching, vertex arrivals	13.5	8
transversal matroid	16	8
graphic matroid	8	4
low density matroid	$4\gamma(M)^3$	$2\gamma(M)$
column k-sparse linear matroid	$4k$	$2k$

Kaplan, Naori, Raz [2022]

Setting	Factor
bipartite matching, vertex arrivals	$3 - 2\sqrt{2} \approx 5.83$
general matching, edge arrivals	13.5

SSPI vs. OOS

Theorem (Informal) [Caramanis et al. [2022]

There is a constant-factor preserving reduction from order-oblivious secretary to “pointwise” single-sample prophet inequalities.*

* Definition related to deferred decision approach.
See paper for formal definition.

Conclusion & Directions

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- Summary of results and techniques for sample-based prophet inequalities
    - Focused on single-choice problem
    - Briefly discussed extensions to combinatorial domains

- Understand boundaries/relative strength of single-sample prophet inequalities
    - $O(1)$ for general matroids?
    - $O(1)$ for submodular/XOS combinatorial auctions?

- Additional technical innovations?

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Thanks! Questions?!